Topic A

The Story of Trigonometry and Its Contexts

F-IF.C.7e, F-TF.A.1, F-TF.A.2

Focus Standards:	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-TF.A.1	Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
	F-TF.A.2	Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
Instructional Days:	10	
Lesson 1:	Ferris Wheels—Tracking the Height of a Passenger Car (E)[1]	
Lesson 2:	The Height and Co-Height Functions of a Ferris Wheel (E)	
Lesson 3:	The Motion of the Moon, Sun, and Stars—Motivating Mathematics (S)	
Lesson 4:	From Circle-ometry to Trigonometry (S)	
Lesson 5:	Extending the Domain of Sine and Cosine to All Real Numbers (S)	
Lesson 6:	Why Call It Tangent? (S)	
Lesson 7:	Secant and the Co-Functions (S)	
Lesson 8:	Graphing the Sine and Cosine Functions (E)	
Lesson 9:	Awkward! Who Chose the Number 360, Anyway? (S)	
Lesson 10:	Basic Trigonometric Identities from Graphs (E)	

In Topic A, students develop an understanding of the six basic trigonometric functions as functions of the amount of rotation of a point on the unit circle and then translate that understanding to the trigonometric functions as functions on the real number line. In Lessons 1 and 2, a Ferris wheel provides a familiar context for the introduction of periodic functions that lead to the sine and cosine functions in Lessons 4 and 5. Lesson 1 is an exploratory lesson in which students model the circular motion of a Ferris wheel using a paper plate. The goal is to study the vertical component of the circular motion with respect to the degrees of rotation of

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

the wheel from the initial position. This function is temporarily described as the *height function* of a passenger car on the Ferris wheel, and students produce a graph of the height function from their model. In this first lesson, students begin to understand the periodicity of the height function as the Ferris wheel completes multiple rotations (MP.7).

Lesson 2 introduces the *co-height function*, which describes the horizontal component of the circular motion of the Ferris wheel. Students again model the position of a car on a rotating Ferris wheel using a paper plate, this time with emphasis on the horizontal motion of the car. In the first lesson, heights were measured from the "ground" to the passenger car of the Ferris wheel, so that the graph of the height function was contained within the first quadrant of the Cartesian plane. In this second lesson, we change our frame of reference so that the values of the height and co-height functions oscillate between $-r$ and r, where r is the radius of the wheel, inching the height and co-height functions toward the sine and cosine functions. The goal of these first two lessons is to provide a familiar context for circular motion so that students can begin to see how the horizontal and vertical components of the position of a point rotating around a circle can be described by periodic functions of the amount of rotation. Reference is made to this context as needed throughout the module.

Lesson 3 provides historical background on the development of the sine and cosine functions in India around 500 C.E. In this lesson, students generate part of a sine table and use it to calculate the positions of the sun in the sky, assuming the historical model of the sun following a circular orbit around Earth. This lesson provides a second example of circular motion that can be modeled using the sine and cosine functions. In this lesson, the link is made between the assumed circular motion of stars and the sun and the periodic sine and cosine functions, and that link is formalized in Lesson 4.

Lesson 4 draws connections between the height function of a Ferris wheel and the sine and cosine functions used in triangle trigonometry in Geometry. This lesson extends the domain of the sine and cosine functions from the restricted domain $(0,90)$ of degree measures of acute angles in triangles to the interval $(0,360)$. Abstracting the sine and cosine from the height and co-height functions of the Ferris wheel allows students to practice MP.2.

In fully developing **F-TF.A.2** on extending the trigonometric functions to the entire real line in Lesson 5, students need to come to know enough values of these functions to generate graphs of these functions and discern structure and properties about them (in much the same way that students were first introduced to exponential functions by studying their values at integer inputs). The most important values to learn, of course, are the values of sine and cosine functions of the most commonly used reference points: the sine and cosine of degree measures that are multiples of $30°$ and $45°$. This knowledge, in turn, serves as concrete examples for learning standard **F-TF.A.1**.

Lessons 6 and 7 introduce the tangent and secant functions through their geometric descriptions on a circle and link those geometric descriptions to the appropriate ratios of sine and cosine. The remaining trigonometric functions, cotangent and cosecant, are also introduced.

In Lesson 8, students construct a graph of the sine and cosine functions as functions on the real line by measuring the horizontal and vertical components of a point on the unit circle, breaking a piece of spaghetti to the appropriate length, and gluing it to the graph. Physically creating the graphs using direct measurement ties together the definition of $\sin(\theta°)$ as the y-coordinate of the point on the unit circle that has been rotated θ degrees about the origin from the point $(1,0)$ and the value of the periodic function $f(\theta) = \sin(\theta°)$.

Lesson 9 introduces radian measure. We justify the switch to radians by drawing the graph of $y = \sin(x°)$ with the same scale on the horizontal and vertical axes, which is nearly impossible to draw. This somewhat artificial task serves many different purposes; it provides justification for the use of radian measures without referring directly to ideas of calculus, it foreshadows the lessons to come in Topic B on transforming the graph of the sine function, and it allows students to look for patterns. Students practice MP.7 when they discover the effects of changing the parameters on the graph, and they practice MP.8 when they repeatedly draw graphs of sinusoidal functions to notice the patterns. Drawing on their experience with graphing parabolas given by $y = kx^2$, students experiment with graphing calculators to produce graphs of $y = \sin(kx°)$ until they find that when $k \approx 57$ (or, equivalently, $k = \frac{180}{\pi}$), the line $y = x$ is tangent to $y = \sin(kx°)$ at the origin. Although we define the sine and cosine functions explicitly as functions of the amount of rotation of the initial ray comprised of the nonnegative part of the x-axis, at the end of Lesson 9 students see that the measure of an angle θ in radians is the length of the arc subtended by the angle as specified by **F-TF.A.1**. Radian measure is used exclusively through the remaining lessons in the module.

The problem set for Lesson 9 focuses on finding the values of the sine and cosine functions for multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, which aligns with **F-TF.A.3**. As students transition to this new way of measuring rotation, these reference points and their trigonometric values help students to make sense of radian measure. The goal of this work, which began in the Geometry course, is for students to fluently and automatically recall (or be able to derive) these values in the Precalculus and Advanced Topics course, thereby satisfying the expectation of **F-TF.A.3**.

The topic culminates with Lesson 10, which incorporates such identities as $\sin(\pi - x) = \sin(x)$ and $\cos(2\pi - x) = \cos(x)$ for all real numbers x into an introduction to trigonometric identities that will be studied further in Topic B. In this lesson, students analyze the graphs of the sine and cosine function and note some basic properties that are apparent from the graphs and from the unit circle, such as the periodicity of sine and cosine, the even and odd properties of the functions, and the fact that the graph of the cosine function is a horizontal shift of the graph of the sine function. Students also note the intercepts and end behavior of these graphs.

Topic B

Understanding Trigonometric Functions and Putting Them to Use

F-IF.C.7e, F-TF.B.5, F-TF.C.8, S-ID.B.6a

Focus Standards:	F-IF.C.7e	Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-TF.B.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.
	F-TF.C.8	Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
	S-ID.B.6a	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*
Instructional Days:	7	
Lesson 11:	Transforming the Graph of the Sine Function (E)[1]	
Lesson 12:	Ferris Wheels—Using Trigonometric Functions to Model Cyclical Behavior (E)	
Lesson 13:	Tides, Sound Waves, and Stock Markets (P)	
Lesson 14:	Graphing the Tangent Function (E)	
Lesson 15:	What Is a Trigonometric Identity? (P)	
Lesson 16:	Proving Trigonometric Identities (P)	
Lesson 17:	Trigonometric Identity Proofs (P)	

In Topic A, students developed the ideas behind the six basic trigonometric functions, focusing primarily on the sine function. In Topic B, students use trigonometric functions to model periodic behavior. We end the module with the study of trigonometric identities and how to prove them.

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 11 continues the idea started in Lesson 9 in which students graphed $y = \sin(kx°)$ for different values of k. In Lesson 11, teams of students work to understand the effect of changing the parameters A, ω, h, and k in the graph of the function $y = A(\sin(\omega(x-h))) + k$, so that in Lesson 12 students can fit sinusoidal functions to given scenarios, which aligns with **F-IF.C.7e** and **F-TF.B.5**. While Lesson 12 requires that students find a formula that precisely models periodic motion in a given scenario, Lesson 13 is distinguished by non-exact modeling, as in **S-ID.B.6a**. In Lesson 13, students analyze given real-world data and fit the data with an appropriate sinusoidal function, providing authentic practice with MP.3 and MP.4 as they debate about appropriate choices of functions and parameters.

Lesson 14 returns to the idea of graphing functions on the real line and producing graphs of $y = \tan(x)$. Students work in groups to produce the graph of one branch of the tangent function by plotting points on a specified interval. The individual graphs are compiled into one classroom graph to emphasize the periodicity and basic properties of the tangent function.

To wrap up the module, students revisit the idea of mathematical proof in Lessons 15–17. Lesson 15 aligns with standard **F-TF.C.8**, proving the Pythagorean identity. In Lesson 17, students discover the formula for $\sin(\alpha + \beta)$ using MP.8, in alignment with standard **F-TF.B.9(+)**, but teachers may choose to present the optional rigorous proof of this formula that is provided in the lesson. Standard **F-TF.B.9(+)** is included because it logically coheres with the rest of the content in the module. Throughout Lessons 15, 16, and 17, the emphasis is on the proper statement of a trigonometric identity as the pairing of a statement that two functions are equivalent on a given domain and an identification of that domain. For example, the identity "$\sin^2(\theta) + \cos^2(\theta) = 1$ for all real numbers θ" is a statement that the two functions $f_1(\theta) = \sin^2(\theta) + \cos^2(\theta)$ and $f_2(\theta) = 1$ have the same value for every real number θ. As students revisit the idea of proof in these lessons, they are prompted to follow the steps of writing a valid proof:

1. Define the variables. For example, "Let θ be any real number."
2. Establish the identity by starting with the expression on one side of the equation and transforming it into the expression on the other side through a sequence of algebraic steps using rules of logic, algebra, and previously established identities. For example:

$$\begin{aligned}\cos(2\theta) &= \cos(\theta + \theta)\\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)\\ &= \cos^2(\theta) - \sin^2(\theta).\end{aligned}$$

3. Conclude the proof by stating the identity in its entirety, both the statement and the domain. For example, "Then, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ for any real number θ."

Lesson 1: Ferris Wheels—Tracking the Height of a Passenger Car

Student Outcomes

- Students apply geometric concepts in modeling situations. Specifically, they find distances between points of a circle and a given line to represent the height above the ground of a passenger car on a Ferris wheel as it is rotated a number of degrees about the origin from an initial reference point.
- Students sketch the graph of a nonlinear relationship between variables.

Lesson Notes

This lesson sets the stage for the study of the sine function by asking students to explore the height of a passenger car on a Ferris wheel at various points on its circular path. The main goal of this first lesson is for students to discover that the relationship between the height and the number of degrees through which the car has rotated from an initial reference position is not linear. In later lessons, students relate this function to the sine function and see how the geometric definitions of sine, cosine, and the other trigonometric ratios can be extended to these circular functions represented in the coordinate plane. In this lesson, in order to separate this new function from the domain-limited sine function that students know from triangle trigonometry, this height function was purposely not referred to as the sine function.

To precisely define and describe the sine and cosine functions in later lessons, the domain must be the real numbers, so that the sine (or cosine) of the number of degrees/radians that an initial ray (i.e., the nonnegative x-axis) has been rotated is taken. In this situation, rotations do not involve any sense of time such as *a rotating clock* or a *turning wheel*. However, it is very natural for students to conceptualize the situation as dependent on time. Because of this conceptual difficulty, old textbooks often defined sine and cosine functions as *spinning a ray about the origin*, where students got a sense of *non-static motion*. Of course, non-static motion is based upon time, which is not how these trigonometric functions are precisely defined. However, the *conceptual image* of the wheel rotating is quite good and important for students to understand; it is just not a definition.

Since the goal of this lesson is not to explain the definition of sine, it is acceptable if students have a *conceptual image* of the wheel spinning with regard to time. However, the stage is also being set for precise definitions of the sine and cosine functions on the real line, and those precise definitions are based on the amount of rotation that has occurred, not on any sense of how much time has elapsed. In Topic B, consideration is given to how to model the motion of the Ferris wheel with respect to time, but until that point, the only concern is with the amount of rotation that a car undergoes from an initial position to a terminal position. Hence, time and motion are not introduced into the first lessons because those lessons are about graphing and plotting specific points (i.e., students measure the degrees of a specific rotation and then measure the specific height associated with it). The motion of the Ferris wheel is explored in a later modeling lesson, but introducing motion now would ultimately distract students from the definitions being developed in these initial lessons.

To further explore the nonlinearity of the Ferris wheel's passenger car height function, students use a paper plate to model a Ferris wheel and actually measure heights at various rotations from an initial reference point. They then create a graph of the paired rotations and heights. During this module, a classroom set of protractors marked in both radians and degrees is needed. Start the module using degrees to measure rotation, and progress in Lesson 9 to using radians to measure rotation. Consistent use of protractors marked with both units eases the transition from the familiar degree measure to the unfamiliar radian measure. A template for these protractors is provided at the end of Lesson 9 and may be printed on transparencies to distribute to students.

Lesson 1 closes by offering a definition of a periodic function and asks students to reflect on why the Ferris wheel height function is an example of a periodic function. In Lesson 2, the paper plate models are again used in creating a graph of both the vertical and horizontal displacements of the car from the axes with respect to the degrees of rotation, eventually leading to the formal definition of sine and cosine functions as the y- and x-coordinates, respectively, of a point on a circle of radius 1 unit for a given number of degrees of rotation θ. Both of these lessons focus significantly on MP.2. Students employ this mathematical practice during many tasks in these two lessons as they relate abstract representations to the movement of the wheel.

Consider splitting this lesson over two class periods to allow more time for students to discuss and share their results with the class. A natural place for this break would be before Exploratory Challenge 2.

Materials

Students model a Ferris wheel using simple, inexpensive supplies. Depending on students, build one class model, have a group or pair of students build a model, or have each student build his own model.

The following tools are needed for the next two lessons.

- Rulers
- Protractors marked in degree and radian units (rotation is measured in degrees in this lesson.)
- Graph paper (optional)
- Colored pencils or pens

The following consumable materials are needed for the next two lessons.

- Small paper plates or 6" circles cut from card stock, 1 per model
- $8.5" \times 11"$ card stock (half of a used manila folder), 1 per model
- A metal brad/fastener (picture shown), 1 per model

Classwork

Opening (5 minutes)

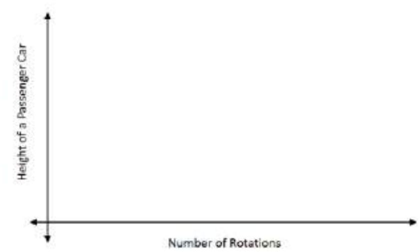
In the last module, students used polynomial and rational functions to model various situations. In this module, functions that represent phenomena that repeat in a predictable way are explored. Show students a video of a Ferris wheel in motion (http://shows.howstuffworks.com/stuff-of-genius/41719-george-ferris-and-his-amazing-wheel-video.htm) or a picture of a Ferris wheel. The link provided above provides a bit of history about the first Ferris wheel, invented by George Ferris in 1893, and also models the motion of the wheel. Ask them to consider how the height of a single passenger car is changing as the car rotates around the wheel. Have them discuss their ideas with a partner, and then have a few students share their thoughts with the whole class. Have students consider other quantities that are changing in this situation such as the horizontal distance from the wheel's center that will become relevant in Lesson 2. The questions below can be used to motivate this discussion. Record student responses on the board or on a sheet of chart paper for reference over the course of Lessons 1 and 2.

- What quantities in this situation are changing as the Ferris wheel rotates?
  - *There are many possible answers, including the height of the cars above the ground, the position of the cars with respect to the ground, the horizontal position of the cars from the center of the wheel, the amount of time that is passing, the amount the rotation changes when the wheel is in motion, and the speed the wheel is rotating.*
- How does the height of a single passenger car change as the car rotates around the wheel?
  - *The height above the ground increases and decreases as the wheel rotates. There is a maximum and minimum height because the cars are positioned on a circle of fixed diameter. After one full $360°$ rotation, a single car is back to the height where it originally started.*

Scaffolding:

- Use questions that proceed from concrete to abstract to support groups having trouble getting started:

  What quantities are going to be included in the sketch? Which one should be the dependent variable, height or amount of rotation? Why does it make sense for height to be the dependent variable?
- Have students create a simple table to help them make their graphs more precise.
- Provide students with a set of axes labeled as shown below.



Exploratory Challenge 1 (5 minutes): The Height of a Ferris Wheel Car

Start students on this problem to help them formalize their thinking from the Opening. The graphs they create will vary widely based on the assumptions that they make. Some groups may want to select a specific height for their Ferris wheel. Some may start the passenger car in different locations. Some may want to consider time as the independent variable along the horizontal axis while others may use the number of turns for the independent variable. The goal of this first exercise is to get a rough sketch on paper and motivate students to want to explore this problem with more precision when they create the paper plate model. Be sure to read through the discussion that follows this problem to understand how to support students during this exploration phase.

Exploratory Challenge 1: The Height of a Ferris Wheel Car

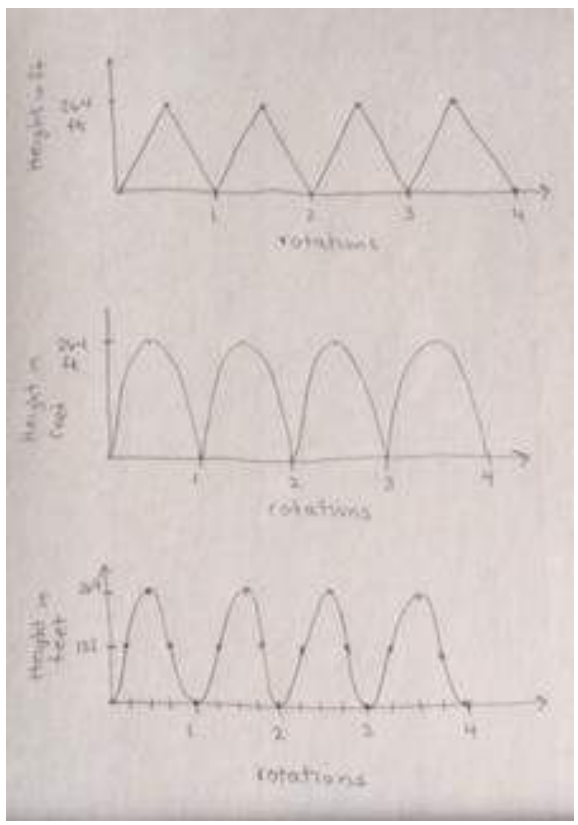
George Ferris built the first Ferris wheel in 1893 for the World's Columbian Exhibition in Chicago. It had 30 passenger cars, was 264 feet tall and rotated once every 9 minutes when all the cars were loaded. The ride cost $\$0.50$.

Source: The New York Times/Redux

a. **Create a sketch of the height of a passenger car on the original Ferris wheel as that car rotates around the wheel 4 times. List any assumptions that you are making as you create your model.**

Three possible solutions are shown. Notice that the third response is closest to an actual sinusoidal graph.

We assumed the car was traveling at a constant speed. We assumed that the car we chose started at the bottom of the wheel. We thought about where the car would be every quarter rotation to help us create the graph. We assumed that we were measuring height above the ground and that the car started at a height of 0 units.



b. **What type of function would best model this situation?**

This answer should be consistent with the sketch in Exercise 1. They might respond that the graph could be line segments, so a piecewise linear function would work, or it could be semicircles since the graph is based on the car moving around a circle.

Discussion (8 minutes)

Have different groups present their results. Start with groups whose graphs are less precise and detailed, and finish with groups whose sketches look most like the graph of a sinusoidal function. Use the questions below to debrief the class sketches. Let the conclusions about the shape of the graph come from the group. Do not worry if groups do not propose a graph that is nonlinear or nonsemicircular.

In this section, guide the discussion around the sketches created in Exploratory Challenge 1 to arrive at two conclusions. These are the most important understandings for students to have at this point in the lesson.

- The height of the car will repeat as the wheel rotates. For any given point on the Ferris wheel, the height of the car at that point will be the same, regardless of how many revolutions the wheel has completed to get to that point, and
- The range of the function represented by this graph is a closed interval whose length corresponds to the diameter of the wheel.

At this point, many students may be graphing a chevron pattern composed of straight line segments or a wave pattern composed of semicircles, while some may have offered a convincing argument that this relationship cannot be linear or a collection of semicircles. The paper plate exploration that follows will help students to make a final decision regarding the shape of the graph. The following questions can be used to focus the discussion of the student graphs. The answers are representative of what students might say in response, but encourage and accept all reasonable answers in this very open-ended discussion.

MP.2 MP.4 & MP.6

- How did you decide where to locate your points on this graph?
  - *We started with the car at ground level and then graphed a point halfway around when it was at the top of the wheel. We also graphed the points when the car was one-fourth and three-fourths of the way around. We decided that the wheel was rotating counterclockwise from our viewpoint.*
- How could you make your assumed model more precise?
  - *We could measure the height at more points during the rotation, such as every one-eighth of a rotation.*
- Is a passenger car ever at the same height during its rotation? How do you know?
  - *Yes, the heights keep repeating each time the wheel rotates. They are also the same height when the car is going up and when it is coming down. For example, at one-fourth and three-fourths of a rotation, the car will be the same height above the ground.*
- Where did you choose to start your car and why?
  - *We started it at the bottom of the wheel because that is where you get on a Ferris wheel.*
- Do you think the height is changing in a linear fashion? Why or why not?
  - *No. When we are at one-fourth of a rotation, the height is half of the wheel's diameter. When we are at three-eighths of a rotation, the height is NOT three-fourths of the wheel's diameter. Therefore, there is no proportional relationship between height and amount of rotation.*
- What patterns did you notice in the way the height of the passenger car was changing?
  - *The height of the car keeps repeating each time the wheel rotates. Pick any point on the wheel, and the car will return to that height after a* $360°$ *rotation.*
- How does the range of the graph of the Ferris wheel height function relate to the physical features of the wheel?
  - *The range is the distance between the maximum and minimum height, and it will always be the diameter of the wheel.*

Exercises 1–5 (5 minutes)

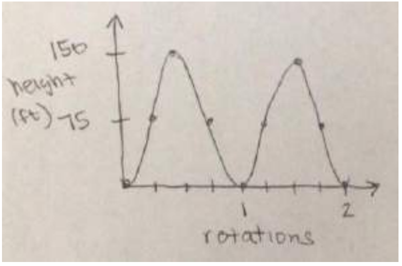
These exercises should be done in small groups or with a partner. Do as many as time permits, especially if the preceding Exploratory Challenge and Discussion exceeded time limits. These exercises provide the opportunity to informally assess how well students processed the preceding discussion. There should be more precise graphs and less linearity (depending on the previous discussion outcomes). Students should begin to understand how changing assumptions about the point from which to measure the height or the starting position of the passenger car will change the appearance of the sketch in a consistent fashion without changing its basic shape. At this point, students may begin to talk about these graphs using the language of transformations, but if they do not, that will come later in the module.

MP.2 & MP.7

Exercises 1–5

1. **Suppose a Ferris wheel has a diameter of 150 feet. From your viewpoint, the Ferris wheel is rotating counterclockwise. We will refer to a rotation through a full $360°$ as a *turn*.**

   a. **Create a sketch of the height of a car that starts at the bottom of the wheel and continues for two turns.**

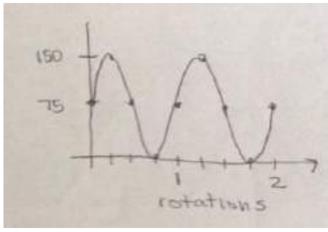


   b. **Explain how the features of your graph relate to this situation.**

   The maximum value is the diameter of the Ferris wheel in feet. The graph repeats because we graphed two turns of the wheel. The minimum values occur when the passenger car is at ground level.

2. **Suppose a Ferris wheel has a diameter of 150 feet. From your viewpoint, the Ferris wheel is rotating counterclockwise.**

   a. **Your friends board the Ferris wheel, and the ride continues boarding passengers. Their car is in the three o'clock position when the ride begins. Create a sketch of the height of your friends' car for two turns.**



   b. **Explain how the features of your graph relate to this situation.**

   The maximum value is the diameter of the Ferris wheel in feet. The graph repeats because we graphed two turns of the wheel. The minimum values occur when the passenger car is at ground level. The vertical intercept of this graph is $(0, 75)$ because we started graphing when the passenger car was at the three o'clock position, which is half the distance to the top of the wheel.

3. **How would your sketch change if the diameter of the wheel changed?**

   The maximum and minimum heights would change. The distance between the maximum and minimum heights is the diameter of the wheel. The sketch would be a vertical stretch or a shrinking of the original sketch, depending on whether or not the diameter was respectively larger or smaller than 150 feet.

4. **If you translated the sketch of your graph down by the radius of the wheel, what would the x-axis represent in this situation?**

   The x-axis would represent a line through the center of the Ferris wheel that is parallel to the ground.

5. **How could we create a more precise sketch?**

   We could measure the height at smaller and smaller increments of rotation.

As groups are working, circulate around the room offering tips, suggestions, and gentle corrections if needed. After students have completed these exercises, invite one or two volunteers to come to the board and share their responses with the class. Before moving on, make sure to include a transition statement that relates to Exercise 5 and that reminds students of the goals of this lesson. For example,

- If we want to create a more accurate graph of the height of a car, then we must measure the height at additional points around the wheel.
- The height of a passenger car is changing in a predictable way as the wheel rotates. Even if we start tracking additional positions of the car on the wheel or change the diameter of the wheel, the graph of this function has a consistent shape and pattern.

Exploratory Challenge 2 (15 minutes): The Paper Plate Model

Introduce this section by reminding students that a physical model and appropriate tools can help them create a more precise graph. This section of the lesson focuses students specifically on making sense and persevering (MP.1), using appropriate tools strategically (MP.5), and attending to precision (MP.6).

Give each group a paper plate, a sheet of card stock or construction paper, and a metal brad fastener. Have them affix the paper plate to the sheet of card stock with a brad located at the center of the plate. Turn the paper plate counterclockwise to model the motion of the Ferris wheel. To create a more accurate graph of the height function, students place marks on the edge of the plate to represent a passenger car every $15°$ around the circle. Then, they measure the height above the ground at every $15°$ for one complete turn, starting with the car in the 3 o'clock position.

To be consistent with the definition of the sine function that will appear in later lessons, $0°$ of rotation is assigned to the 3 o'clock position of the car, and forward motion of the wheel is modeled by counterclockwise rotation of the paper plate. Informally define rotations greater than $180°$ for students. Most students are familiar with extreme sports such

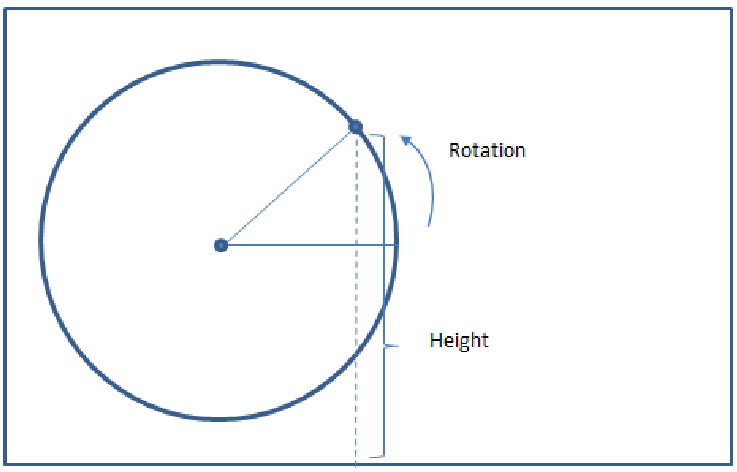
Scaffolding:

- In order to keep this exploration even more open, choose to just give students the materials, tools, and a sheet of graph paper with the directions to create a more accurate graph of the Ferris wheel height, and then offer suggestions and support while circulating among the groups, or have them work using the supports provided in the student materials.
- As an alternative to using a protractor to measure the rotation on the paper plate, students can use a compass to create a circle on a piece of paper and construct $15°$ arcs along its circumference. One method to create these arcs would be to section the circle into $60°$ arcs and then bisect the corresponding central angles into $30°$ and then $15°$ angles.
- Help students who struggle with measuring using a ruler and protractor by modeling a few measurements and table entries on a document camera or the board.

as skateboarding or snowboarding and are comfortable referring to one complete turn as a 360. Also, explain that a $270°$ rotation is simply a $180°$ rotation followed by a $90°$ rotation. Similarly, any rotation between $180°$ and $360°$ is simply the sequence of a $180°$ rotation followed by a rotation between $0°$ and $180°$. Have students record their measurements in the table below on the student pages and then create a graph of the ordered pairs in the table.

The diagram below shows what the paper plate will look like when it is mounted on the card stock and indicates the rotation and corresponding height students should be measuring. Notice that this diagram will be consistent with the formal definition of sine as the y-coordinate of a point on the unit circle where the terminal ray intersects the circle. Let the lower edge of the paper represent ground level. It does not matter where students affix the paper plate to the paper. It will only affect the height above the horizontal axis of their graphs.

However, students should try to locate the center of the plate as precisely as possible and mark the $15°$ intervals around the edge of the plate as accurately as possible as well. Let students struggle with both of these measurement challenges. Some may locate the center by folding their plate into quarters; others may use a ruler to help locate the center by drawing two diameters. It may be helpful to have students label one radius (shown below with a solid line) that they will keep horizontal as they mark the $15°$ intervals for the rotations from the initial reference position on the edge of the plate. It may also help them to label the measurements directly on the plate that correspond to the numbers in the table.



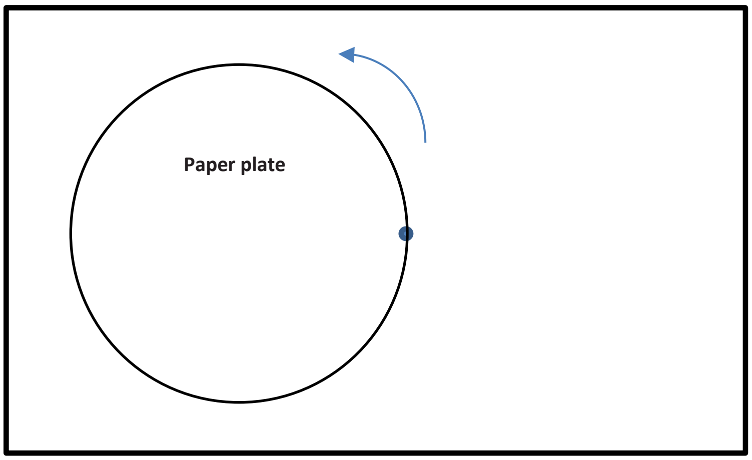
Students work in small groups to build a physical model and measure the amount of rotation and corresponding heights. The student pages provide scaffolds, including a diagram they can mark up to help them understand how to measure the heights and rotation, and a table to record their measurements. A grid has been provided for their graph as well, but students should provide appropriate labels for the axes.

After measuring and constructing a graph, students' original ideas about the height function will be either refuted or verified.

Exploratory Challenge 2: The Paper Plate Model

Use a paper plate mounted on a sheet of paper to model a Ferris wheel, where the lower edge of the paper represents the ground. Use a ruler and protractor to measure the height of a Ferris wheel car above the ground for various amounts of rotation. Suppose that your friends board the Ferris wheel near the end of the boarding period, and the ride begins when their car is in the three o'clock position as shown.

a. **Mark the diagram below to estimate the location of the Ferris wheel passenger car every 15 degrees. The point on the circle below represents the passenger car in the 3 o'clock position. Since this is the beginning of the ride, consider this position to be the result of rotating by $0°$.**



b. **Using the physical model you created with your group, record your measurements in the table, and then graph the ordered pairs (rotation, height) on the coordinate grid shown below. Provide appropriate labels on the axes.**

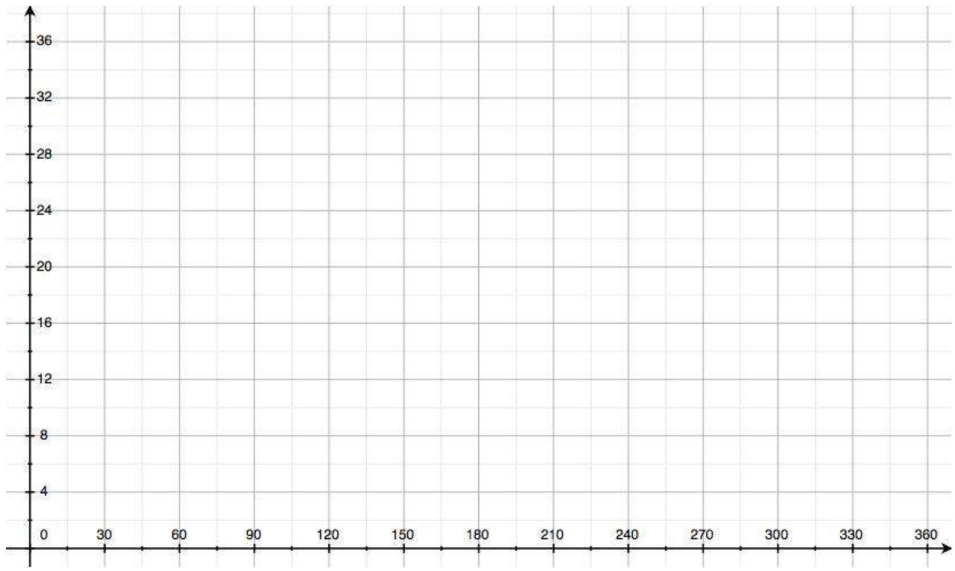
Rotation (degrees)	Height (cm)
0	
15	
30	
45	
60	
75	
90	

Rotation (degrees)	Height (cm)
105	
120	
135	
150	
165	
180	
195	

Rotation (degrees)	Height (cm)
210	
225	
240	
255	
270	
285	
300	

Rotation (degrees)	Height (cm)
315	
330	
345	
360	

Height as a Function of Degrees of Rotation



c. **Explain how the features of your graph relate to the paper plate model you created.**

The domain values represent the rotation measured in degrees of a car on the Ferris wheel from the 3 o'clock position for one turn of a passenger car. The range values represent the height of the car above the ground, which we modeled using the bottom edge of the paper. The vertical distance between the maximum and minimum values is the diameter of the paper plate measured in centimeters.

Display several graphs, and discuss their similarities and differences. Discuss the challenges in this lesson to use tools strategically and attend to precision when measuring. Also, address any differences in the maximum and minimum values of students' graphs. Make sure to connect these differences to the location of the center of the circle above the bottom edge of the paper.

MP.2

Encourage quantitative reasoning by asking students to relate features of the graph to the rotating Ferris wheel. These questions can guide that discussion.

- How can you identify the diameter of the Ferris wheel from your graph?
  - *It is the vertical distance between the highest and lowest points on the graph.*
- If the paper plate model was scaled so that 1 cm on the plate represented 5 ft. on a real Ferris wheel, what is the diameter of the wheel?
  - *Answers will vary depending on plate size, but they should be* 5 *times the actual diameter. So, a plate with a* 20 cm *diameter would represent a* 100 ft. *diameter Ferris wheel.*
- How high above the ground is the lowest point on the Ferris wheel?
  - *Answers will vary but should correspond to the second coordinate of the minimum point on the graph.*
- Why isn't the diameter of the Ferris wheel the same as the maximum value on your graph? (*Note that some students might have actually placed their plate so that it touches the lower edge of the paper, in which case, this question does not apply.)*
  - *It is not the same because we positioned the bottom of our paper plate model above the edge of the paper which represented ground level, a height of* 0 ft.

Closing (3 minutes)

Share the definition of a periodic function to bring closure to this lesson. A formal definition appears below. Consider sharing a more student-friendly version with your students. Using a vocabulary organizer like a Frayer diagram helps students make sense of this important vocabulary word and encourages students to rephrase the meaning in their own words, provide examples and non-examples, and create a visual representation of the word. Periodicity is one of the fundamental ideas of this module, so revisit the term *periodic function* often and include it on a vocabulary word wall.

> **PERIODIC FUNCTION:** A function f whose domain is a subset of the real numbers is said to be *periodic with period* $P > 0$ if the domain of f contains $x + P$ whenever it contains x, and if $f(x + P) = f(x)$ for all real numbers x in its domain. If a least positive number P exists that satisfies this equation, it is called the *fundamental period*, or if the context is clear, just the *period* of the function.

Discuss how the height of a rotating Ferris wheel can be represented by a periodic function if we extend the domain to represent multiple turns. Ask students to identify the fundamental period P of the height function of the Ferris wheel. Remind students that for any point on the Ferris wheel, the passenger car will return to that same height after one full turn of $360°$, so if we measure rotation in degrees, the fundamental period of the height function is 360. If we measure rotation as fractions of a turn, the fundamental period of the height function is 1.

Have students respond to the following questions in writing or with a partner.

Closing

- **How does a function like the one that represents the height of a passenger car on a Ferris wheel differ from other types of functions you have studied such as linear, polynomial, and exponential functions?**
- **What is the domain of your Ferris wheel height function? What is the range?**
- **Provide a definition of periodic function in your own words. Why is the Ferris wheel height function an example of a periodic function?**
- **What other situations might be modeled by a periodic function?**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 1: Ferris Wheels—Tracking the Height of a Passenger Car

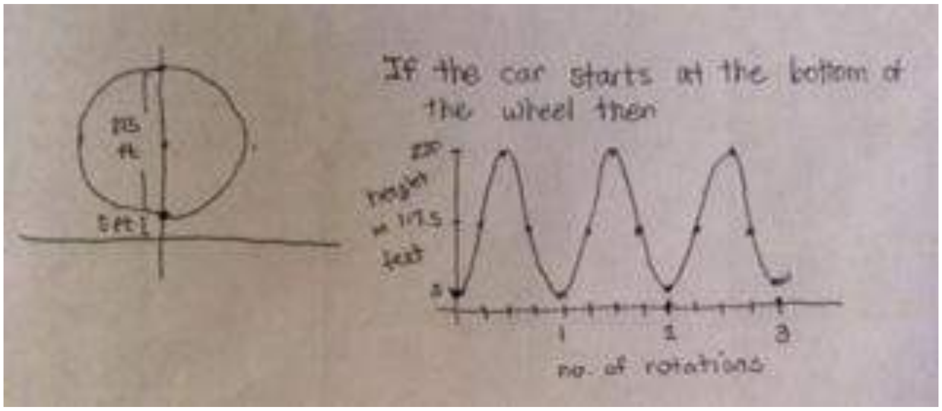
Exit Ticket

1. Create a graph of a function that represents the height above the ground of the passenger car for a 225-foot diameter Ferris wheel that completes three turns. Assume passengers board at the bottom of the wheel, which is 5 feet above the ground, and that the ride begins immediately afterward. Provide appropriate labels on the axes.

2. Explain how the features of your graph relate to this situation.

Exit Ticket Sample Solutions

1. **Create a graph of a function that represents the height above the ground of the passenger car for a 225-foot diameter Ferris wheel that completes three turns. Assume passengers board at the bottom of the wheel, which is 5 feet above the ground, and that the ride begins immediately afterward. Provide appropriate labels on the axes.**



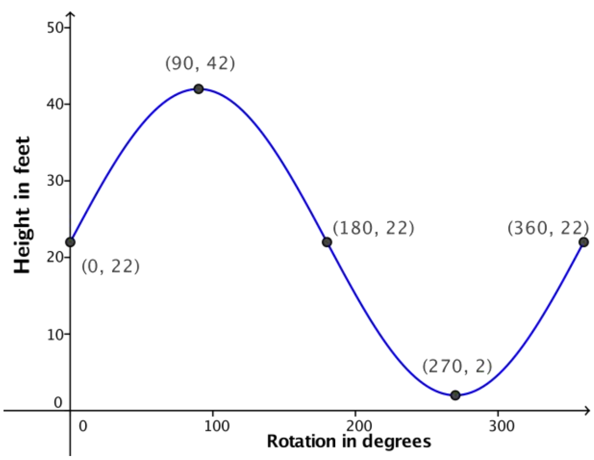
2. **Explain how the features of your graph relate to this situation.**

   The first maximum point on the graph is $(180, 230)$. This shows the height of a passenger car above the ground after half a turn. The car will reach this point again after one and a half and two and a half turns. The first minimum point of the graph is $(0, 5)$. This point represents the height of the passenger car at the bottom of the wheel, and this is where we started the graph. The difference between the maximum and minimum y-coordinates is the diameter of the wheel. When the function increases, the car is rising, and when it decreases, the car is moving back down.

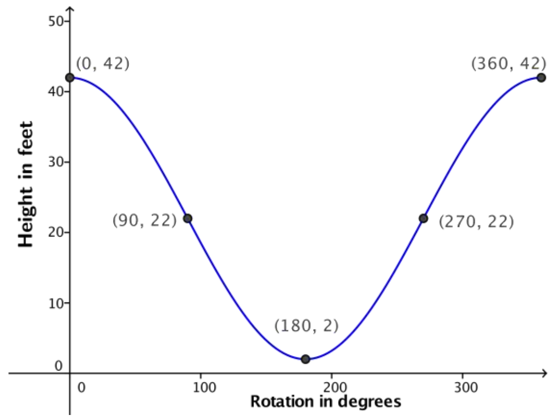
Problem Set Sample Solutions

1. **Suppose that a Ferris wheel is 40 feet in diameter and rotates counterclockwise. When a passenger car is at the bottom of the wheel, it is located 2 feet above the ground.**

   a. **Sketch a graph of a function that represents the height of a passenger car that starts at the 3 o'clock position on the wheel for one turn.**



b. **Sketch a graph of a function that represents the height of a passenger car that starts at the top of the wheel for one turn.**



c. **The sketch you created in part (a) represents a graph of a function. What is the domain of the function? What is the range?**

The domain is $[0, 360]$. The range is $[2, 42]$.

d. **The sketch you created in part (b) represents a graph of a function. What is the domain of the function? What is the range?**

The domain is $[0, 360]$. The range is $[2, 42]$.

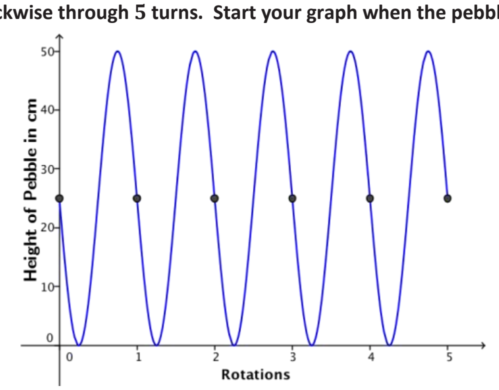
e. **Describe how the graph of the function in part (a) would change if you sketched the graph for two turns.**

The graph would repeat itself over another interval of 360 degrees. It would appear as if we attached a horizontal translation of the graph to the right end of the original graph.

f. **Describe how the function in part (a) and its graph would change if the Ferris wheel had a diameter of 60 feet.**

The function's range would change to $[2, 62]$, and the graph's maximum value would be 62 instead of 42. The domain would stay the same, and the x-coordinates where the maximum and minimum values occurred would not change.

2. **A small pebble is lodged in the tread of a tire with radius 25 cm. Sketch the height of the pebble above the ground as the tire rotates counterclockwise through 5 turns. Start your graph when the pebble is at the 9 o'clock position.**



3. The graph you created in Problem 2 represents a function.

a. Describe how the function and its graph would change if the tire's radius was 24 inches instead of 25 cm.

The highest y-value on the graph would be 48 inches instead of 50 centimeters. The function's range would change to $[0, 48]$.

b. Describe how the function and its graph would change if the wheel was turning in the opposite direction.

The graph would increase to a height of 50 cm after $\frac{1}{4}$ of a turn and then decrease to a height of 0 after $\frac{3}{4}$ of a turn. While the domain and range of the function would remain the same, the actual correspondence between rotations and height would change since the direction of rotation is different.

c. Describe how the function and its graph would change if we started the graph when the pebble was at ground level.

Assuming a counterclockwise rotation, a rotation of 0 degrees would be paired with a height of 0, and then after $\frac{1}{4}$ of a turn, the height would be 25 cm. The domain and range of the function would not change, but the coordinates of the intercepts and maximum and minimum points would change.

4. Justice believes that the height of a Ferris wheel passenger car is best modeled with a piecewise linear function. Make a convincing argument why a piecewise linear function is NOT a good model for the height of a car on a rotating Ferris wheel.

When we measured the height for every 15 degrees of rotation, the height was changing more rapidly at the far left and right positions of the wheel and more slowly near the top and bottom of the wheel. Equal increments of rotation did not result in a constant change in the height. If the function was piecewise linear, then equal changes in the rotation should have corresponded to equal changes in height in places.

Lesson 2: The Height and Co-Height Functions of a Ferris Wheel

Student Outcomes

- Students model and graph two functions given by the location of a passenger car on a Ferris wheel as it is rotated a number of degrees about the origin from an initial reference position.

Lesson Notes

Students extend their work with the function that represents the height of a passenger car on a Ferris wheel from Lesson 1 to define a function that represents the horizontal displacement of the car from the center of the wheel, which is temporarily called the *co-height* function. In later lessons, the co-height function is related to the cosine function. Students sketch graphs of various co-height functions and notice that these graphs are nonlinear. Students explain why the graph of the co-height function is a horizontal translation of the graph of the height function and sketch graphs that model the position of a passenger car for various-sized Ferris wheels. The work in the first three lessons of Module 2 serves to ground students in circular motion and set the stage for a formal definition of the sine and cosine functions in Lessons 4 and 5.

In Lesson 1, students measured the height of a passenger car of the Ferris wheel in relation to the ground and started tracking cars as passengers boarded at the bottom of the wheel. In this lesson, we change our point of view to measure height is measured as vertical displacement from the center of the wheel, and the co-height is measured as the horizontal displacement from the center of the wheel. Additionally, although it is not realistic, cars are tracked rotating around the wheel beginning from the 3 o'clock position. With these changes in perspective, the functions used to model the height and co-height functions are much closer to the basic sine and cosine functions that will be defined in Lesson 4.

In the Exploratory Challenge, students reuse the paper plate model from Lesson 1.

Classwork

Opening Exercise (5 minutes)

Ask students to recall the quantities that change as a passenger car moves around a Ferris wheel. These were discussed and recorded in the opening discussion of Lesson 1. In this lesson, both the vertical position of the passenger car and the horizontal position of the car as the wheel rotates are modeled.

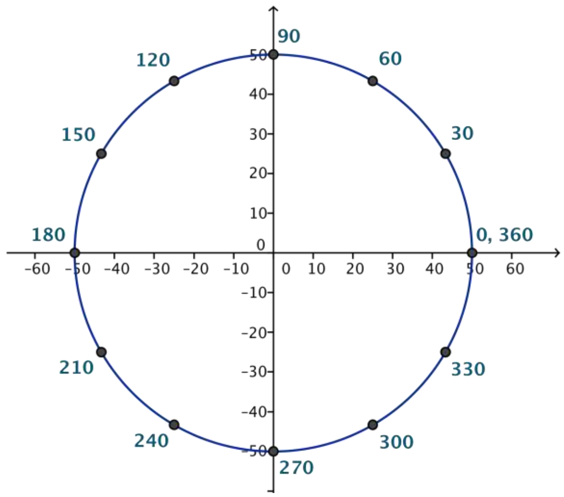
In this lesson, the perspective is changed so as to measure the height of the passenger car on the Ferris wheel from the horizontal line through the center of the wheel. This means that if a Ferris wheel has a radius of 50 feet, then the maximum value of the height function will be 50, and the minimum value of the height function will be -50. In preparation for the introduction of the actual sine and cosine functions in Lesson 4, the 3 o'clock position will consistently be considered the position at which the passengers board the Ferris wheel. Allow students to work in pairs or small groups on this exercise to ensure that all students understand this shift in how the heights are measured. After students have completed the exercise, call for a few volunteers to show their sketches and to explain their reasoning.

MP.4

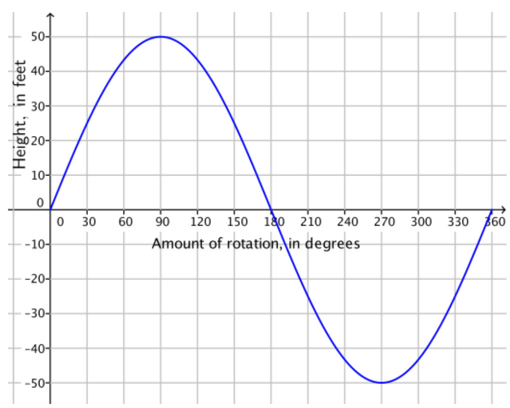
Opening Exercise

Suppose a Ferris wheel has a radius of 50 feet. We will measure the height of a passenger car that starts in the 3 o'clock position with respect to the horizontal line through the center of the wheel. That is, we consider the height of the passenger car at the outset of the problem (that is, after a $0°$ rotation) to be 0 feet.

a. **Mark the diagram to show the position of a passenger car at 30-degree intervals as it rotates counterclockwise around the Ferris wheel.**



b. **Sketch the graph of the height function of the passenger car for one turn of the wheel. Provide appropriate labels on the axes.**



c. **Explain how you can identify the radius of the wheel from the graph in part (b).**

The graph of the height function for one complete turn shows a maximum height of 50 feet and a minimum height of -50 feet, suggesting that the wheel's diameter is 100 feet and thus its radius is 50 feet.

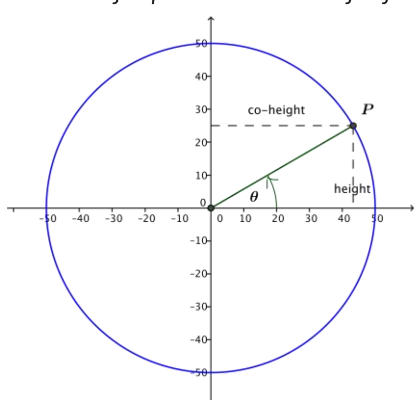
d. **If the center of the wheel is 55 feet above the ground, how high is the passenger car above the ground when it is at the top of the wheel?**

The passenger car is 105 feet above the ground when it is at the top of the wheel. Since the graph displays the height above the center of the wheel, we would need to add 55 feet to 50 feet to get the height (in feet) above the ground.

Discussion (8 minutes)

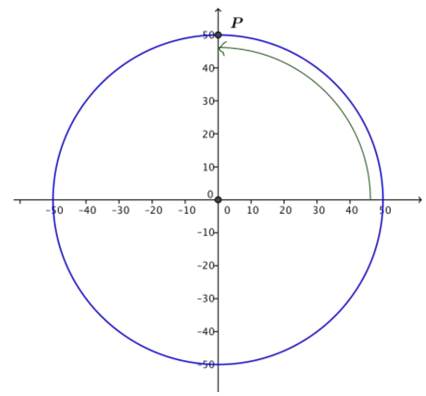
In Lesson 1 and in the Opening Exercise of this lesson, students modeled the height of a passenger car of a Ferris wheel, which is now considered to be the vertical displacement of the car with respect to a horizontal line through the center of the wheel. Now the horizontal position of the cars as they rotate around the wheel, which defines a function called the *co-height* of the passenger car, will be considered.

- Recall that we modeled the height of a passenger car as a function of degrees as the car rotated counterclockwise from the car's starting point at a certain point on the wheel—either the bottom of the wheel or at the 3 o'clock position. Is there another measurement that we can model as a function of degrees rotated counterclockwise from the car's starting position?
  - *The horizontal position of the passenger cars*
- In the Opening Exercise, we changed how we measure the height of a passenger car on the Ferris wheel, and we now consider the height to be the vertical displacement from the center of the wheel. Points near the top of the wheel have a positive height, and points near the bottom have a negative height. That is, we measure the height as the vertical distance from a horizontal line through the center of the wheel. With this in mind, how should we measure the horizontal distance?
  - *We can measure the horizontal displacement from the vertical line through the center of the wheel.*
- We will refer to the horizontal displacement of a passenger car from the vertical line through the center of the wheel as the *co-height* of the car.
- Where is the car when the co-height is zero?
  - *The car is along the vertical line through the center of the wheel, so it is either at the top or the bottom of the wheel.*
- How can we assign positive and negative values to the co-height?
  - *Assign a positive value for positions on the right of the vertical line through the center of the wheel, and assign a negative value for positions on the left of this line.*



> *Scaffolding:*
> Have students record on chart paper the co-height and height of the car's position when it is either on the horizontal or vertical axes and the number of degrees the car has rotated from its initial position at 3 o'clock. Post this chart for quick reference.

- Using our Opening Exercise, what is the starting value of the co-height?
  - *Since the radius of the wheel is* 50 *feet, then the initial co-height at the 3 o'clock position is* 50.



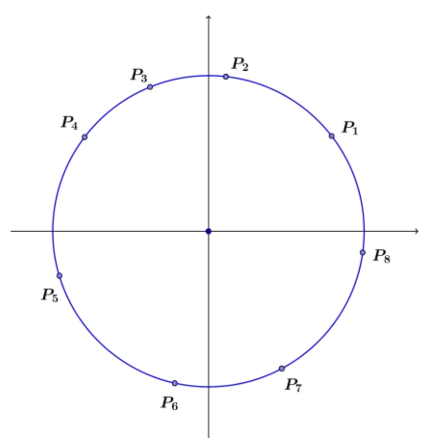
- Now suppose, that the passenger car has rotated 90 degrees counterclockwise from its initial position of 3 o'clock on the wheel. What is the co-height of the car in this position?
  - *After rotating by 90 degrees counterclockwise, the car is positioned at the top of the wheel, so it lies along the vertical line through the center of the wheel. Thus, the co-height is 0 feet.*
- Is there a maximum value of the co-height of a passenger car? Is there a minimum value of the co-height?
  - *When the car is at the 3 o'clock position, the co-height is equal to the radius of the wheel, which is the furthest horizontal position from the vertical axis on the positive side. So for our example, the maximum value of the co-height is 50.*
  - *When the car has rotated 180 degrees from its original position and is located at the 9 o'clock position, the co-height is equal to the opposite of the radius. This is its minimum value. For our example, the minimum value of the co-height is -50.*

Exercises 1–3 (5 minutes)

These exercises can either be completed alone or in pairs. They will provide the opportunity to informally assess how well students understood the preceding discussion. After a few minutes, call on volunteers to share their answers with the class.

Exercises 1–3

1. Each point $P_1, P_2, \ldots P_8$ on the circle in the diagram to the right represents a passenger car on a Ferris wheel.



a. Draw segments that represent the co-height of each car. Which cars have a positive co-height? Which cars have a negative co-height?

The cars corresponding to points P_1, P_2, P_7, and P_8 have a positive co-height. The cars corresponding to points P_3, P_4, P_5, and P_6 have a negative co-height.

b. List the points in order of increasing co-height; that is, list the point with the smallest co-height first and the point with the largest co-height last.

$P_5, P_4, P_3, P_6, P_2, P_7, P_1, P_8$

2. Suppose that the radius of a Ferris wheel is 100 feet and the wheel rotates counterclockwise through one turn. Define a function that measures the co-height of a passenger car as a function of the degrees of rotation from the initial 3 o'clock position.

a. What is the domain of the co-height function?

The domain of the co-height function is $[0, 360]$, where we are measuring in terms of degrees of rotation.

b. **What is the range of the co-height function?**

Because the radius is 100 ft. the range of the co-height function is $[-100, 100]$.

c. **How does changing the wheel's radius affect the domain and range of the co-height function?**

Changing the radius does not change the domain of the co-height function.

The range of the co-height function depends on the radius; for a wheel of radius r, the range of the co-height function is $[-r, r]$.

3. **For a Ferris wheel of radius 100 feet going through one turn, how do the domain and range of the height function compare to the domain and range of the co-height function? Is this true for any Ferris wheel?**

The domain for each function is $[0, 360]$, where rotations are measured in degrees. The range of each function is $[-100, 100]$. For any Ferris wheel, the domain of the height and co-height functions is $[0, 360]$. The range depends on the radius, r, of the wheel, but for both the height and co-height functions, the range is $[-r, r]$. Thus, the height and co-height functions for a Ferris wheel have the same domain and range.

Exploratory Challenge (20 minutes): The Paper Plate Model, Revisited

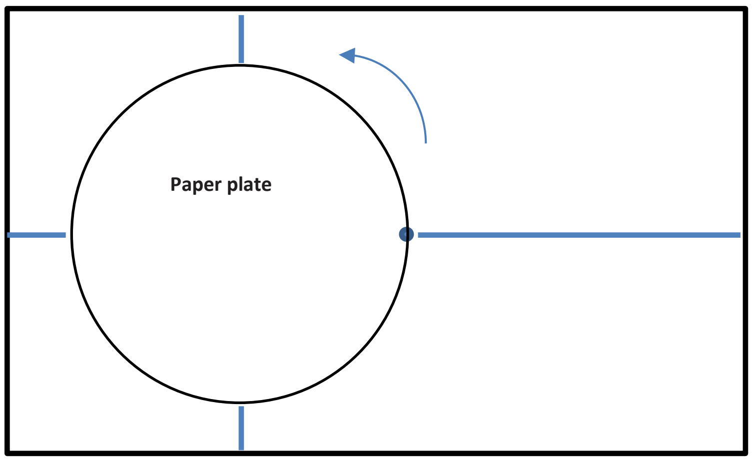
Have students reconvene with the members of their paper plate model groups from Exploratory Challenge 2 in Lesson 1. Redistribute each group's paper plate model, which was submitted at the conclusion of the previous lesson. In Lesson 1, students modeled a passenger car's height relative to the ground (i.e., from the bottom of the paper). So that the models align with the sine and cosine functions that will be introduced in future lessons, have students measure a passenger car's height and co-height relative to the horizontal and vertical axes through the center of the wheel. Instruct students to measure the height and co-height every 15 degrees for a complete turn using their paper plate model. It may be necessary to remind students that the Ferris wheel's motion is counterclockwise. Monitor groups to make sure they are measuring from the axes through the center of the wheel. Students may also need to be reminded that the coordinate system has been set up so that locations below the horizontal axis through the center of the wheel have negative height values, and values left of the vertical axis through the center of the wheel have negative co-height values.

Students work in small groups to build a physical model and measure angles, heights, and co-heights. The student pages provide scaffolds including a diagram they can mark up to help them understand how to measure the heights, co-heights, and angles, as well as a table to record their measurements. Students should record their measurements in the table, and then they should graph the height and co-height functions separately on the axes below, providing appropriate labels on the axes.

Exploratory Challenge: The Paper Plate Model, Revisited

Use a paper plate mounted on a sheet of paper to model a Ferris wheel, where the lower edge of the paper represents the ground. Use a ruler and protractor to measure the height and co-height of a Ferris wheel car at various amounts of rotation, measured with respect to the horizontal and vertical lines through the center of the wheel. Suppose that your friends board the Ferris wheel near the end of the boarding period, and the ride begins when their car is in the three o'clock position as shown.

a. Mark horizontal and vertical lines through the center of the wheel on the card stock behind the plate as shown. We will measure the height and co-height as the displacement from the horizontal and vertical lines through the center of the plate.



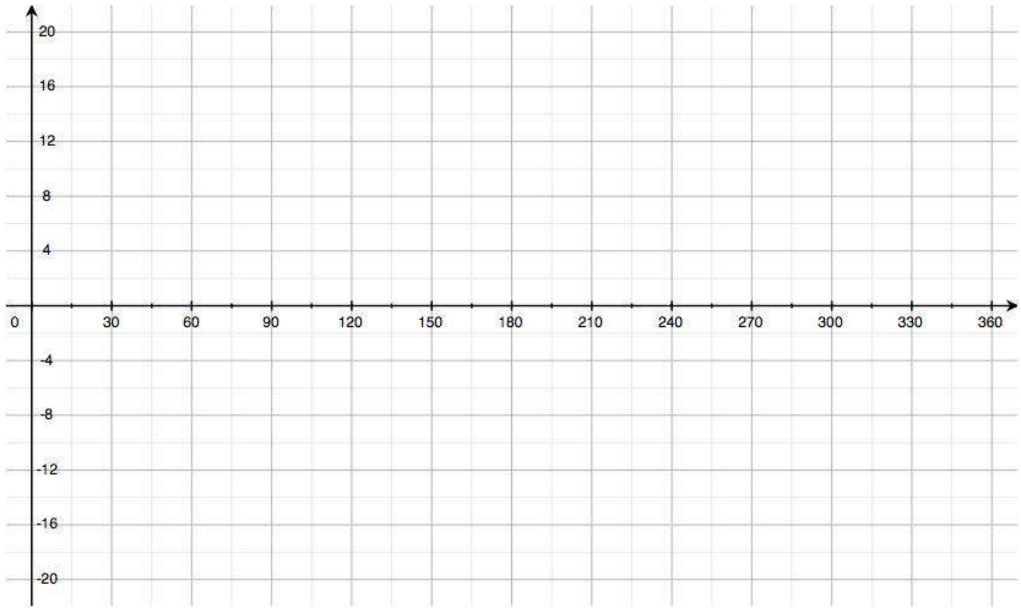
b. Using the physical model you created with your group, record your measurements in the table, and then graph each of the two sets of ordered pairs (rotation angle, height) and (rotation angle, co-height) on separate coordinate grids. Provide appropriate labels on the axes.

Rotation (degrees)	Height (cm)	Co-Height (cm)
0		
15		
30		
45		
60		
75		
90		
105		
120		

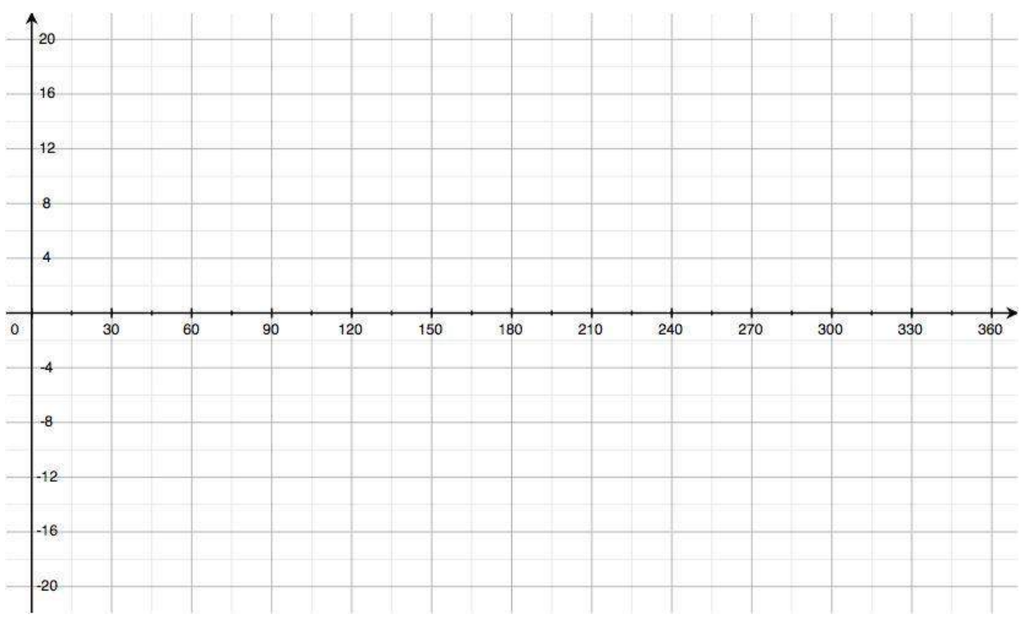
Rotation (degrees)	Height (cm)	Co-Height (cm)
135		
150		
165		
180		
195		
210		
225		
240		

Rotation (degrees)	Height (cm)	Co-Height (cm)
255		
270		
285		
300		
315		
330		
345		
360		

Height as a Function of Degrees of Rotation



Co-Height as a Function of Degrees of Rotation



While graphs may vary slightly from one group to the next, lead students to verbalize that it appears that the co-height graph is a horizontal translation of the height graph (and vice versa).

Encourage quantitative reasoning by asking students to relate features of the graph to the scenario of a car rotating around a Ferris wheel. The following questions can guide that discussion.

- What do the zeros of the graph of the co-height function represent in this situation?
  - *They represent the numbers of degrees of rotation where the passenger car is on the vertical line through the center of the wheel and has a horizontal distance from the center equal to 0. These are the highest and lowest positions of the car during the ride.*
- What does the vertical intercept of the graph of the co-height function represent in this situation?
  - *It represents the radius of the wheel. At the outset of the ride, the car is at the 3 o'clock position, so it has rotated by 0 degrees, and the distance from the center is equal to the radius of the wheel.*
- How are the graphs of the height and co-height functions related to each other?
  - *It looks like one graph is a horizontal translation of the other by $90°$.*

Closing (2 minutes)

Students should respond to these questions in writing or with a partner. Use this as an opportunity to informally assess their understanding of the height and co-height functions.

Closing

- **Why do you think we named the new function the co-height?**

  Both functions measure a distance from the passenger car to one of the axes at various numbers of degrees of rotation of the wheel, so the horizontal measurements are closely related to the vertical measurements.

- **How are the graphs of these two functions alike? How are they different?**

  The graph of the co-height function appears to be a horizontal translation of the graph of the height function. Assuming we create both functions from the same initial passenger car position on the same Ferris wheel, the two functions will have the same domain and the same range, but the values of the functions are not the same for the same amount of rotation. When one function has a value of zero, the other has either a maximum value of 1 or a minimum value of -1.

- **What does a negative value of the height function tell us about the location of the passenger car at various positions around a Ferris wheel? What about a negative value of the co-height function?**

  A negative value of the height function tells us the passenger car is below the center of the Ferris wheel. A negative value of the co-height function tells us the passenger car is left of the center of the Ferris wheel.

Exit Ticket (5 minutes)

Name ________________________________ Date ________________

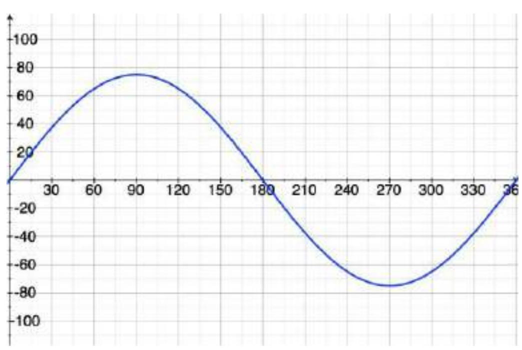
Lesson 2: The Height and Co-Height Functions of a Ferris Wheel

Exit Ticket

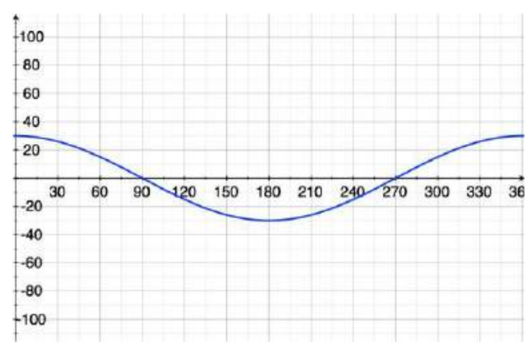
Zeke Memorial Park has two different-sized Ferris wheels, one with a radius of 75 feet and one with a radius of 30 feet. For either wheel, riders board at the 3 o'clock position. Indicate which graph (a)–(d) represents the following functions for the larger and the smaller Ferris wheels. Explain your reasoning.

	Wheel with 75-foot radius		Wheel with 30-foot radius
Height function:	________	Height function:	________
Co-Height function:	________	Co-Height function:	________

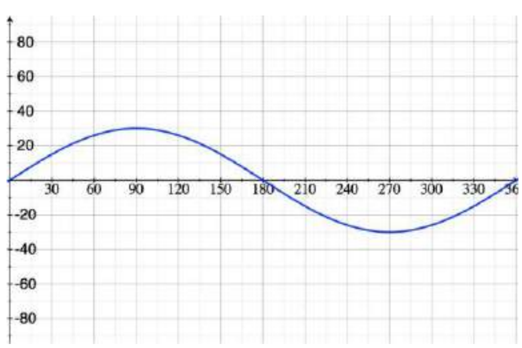
(a)



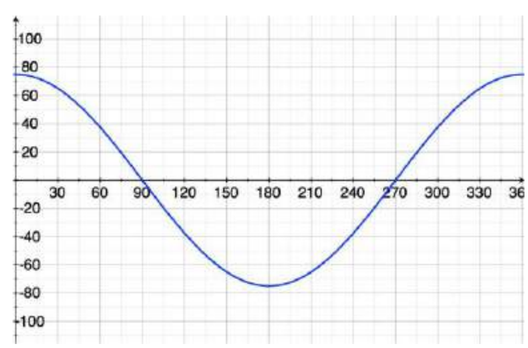
(b)



(c)



(d)

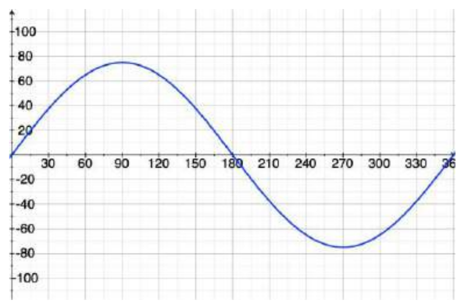


Exit Ticket Sample Solutions

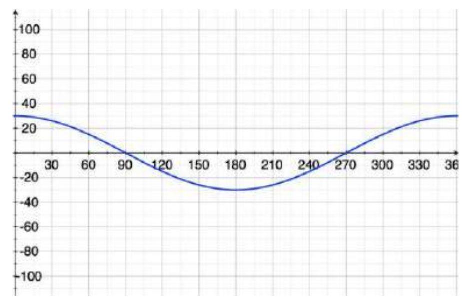
Zeke Memorial Park has two different-sized Ferris wheels, one with a radius of 75 feet and one with a radius of 30 feet. For either wheel, riders board at the 3 o'clock position. Indicate which graphs (a)–(d) represent the following functions for the larger and the smaller Ferris wheels. Explain your reasoning.

	Wheel with 75-foot radius		Wheel with 30-foot radius
Height function:	*(a)*	**Height function:**	*(c)*
Co-Height function:	*(d)*	**Co-Height function:**	*(b)*

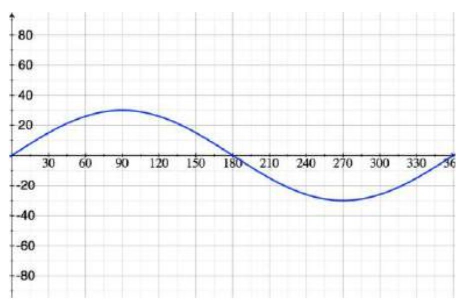
(a)



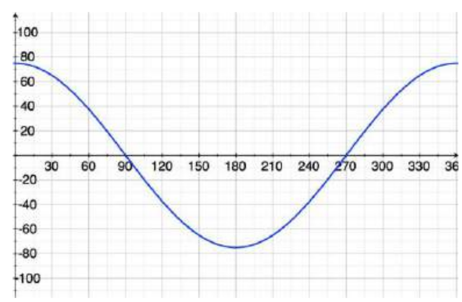
(b)



(c)



(d)



The maximum value of a passenger car's height function over one turn will correspond to the highest point on the wheel, which means that the maximum value of the function is the radius of the wheel. Thus, the graphs that have a maximum value of 75 correspond to the larger Ferris wheel, and the graphs that have a maximum value of 30 correspond to the smaller wheel. Since the cars begin at the 3 o'clock position, the height graphs begin at height zero, while the co-height graphs begin with an initial co-height equal to the radius. Thus, graphs (a) and (d) correspond to the larger wheel, and graphs (b) and (c) correspond to the smaller wheel.

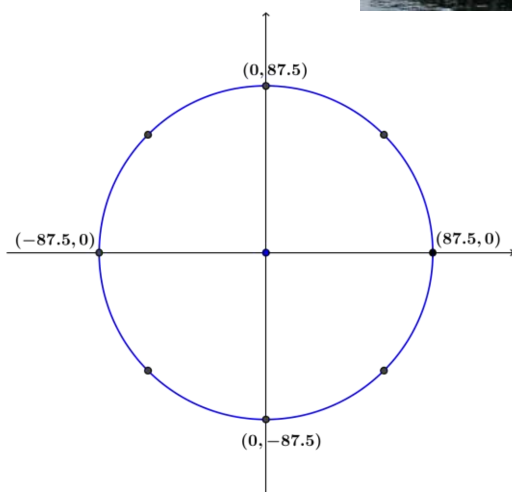
Problem Set Sample Solutions

This Problem Set asks students to confirm their understanding of the co-height function and its relationship to the height function.

1. **The Seattle Great Wheel, with an overall height of 175 feet, was the tallest Ferris wheel on the West Coast at the time of its construction in 2012. For this problem, assume that the diameter of the wheel is 175 feet.**

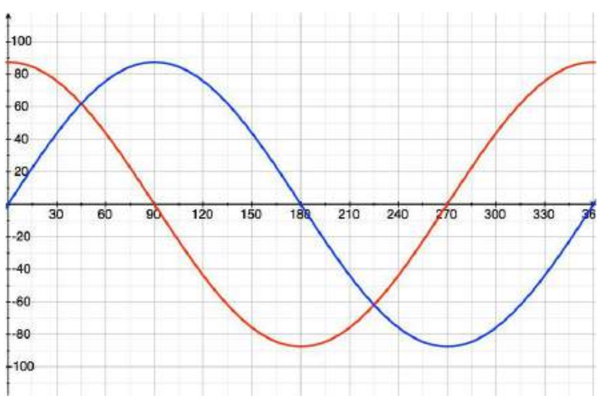
   a. **Create a diagram that shows the position of a passenger car on the Great Wheel as it rotates counterclockwise at 45-degree intervals.**

   The Great Wheel has a diameter of 175 feet, so the radius is 87.5 feet.



   b. **On the same set of axes, sketch graphs of the height and co-height functions for a passenger car starting at the 3 o'clock position on the Great Wheel and completing one turn.**

   Below, the blue curve represents the height function, and the red curve represents the co-height function.



c. **Discuss the similarities and differences between the graph of the height function and the graph of the co-height function.**

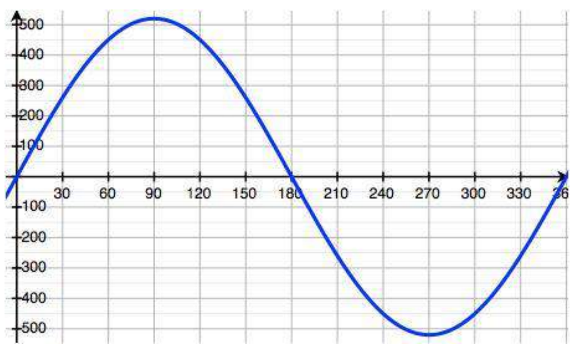
Both the height and co-height functions have the same domain, $[0, 360]$, and range, $[-87.5, 87.5]$. Both functions have the same maximum value of 87.5 and minimum value of -87.5, but they occur at different amounts of rotation. When one function takes on a value of zero, the other either takes on its maximum value of 87.5 or its minimum value of -87.5. The co-height function starts at its maximum value, and the height function starts at zero. The graph of the co-height function is the graph of the height function translated horizontally to the left by 90.

d. **Explain how you can identify the radius of the wheel from either graph.**

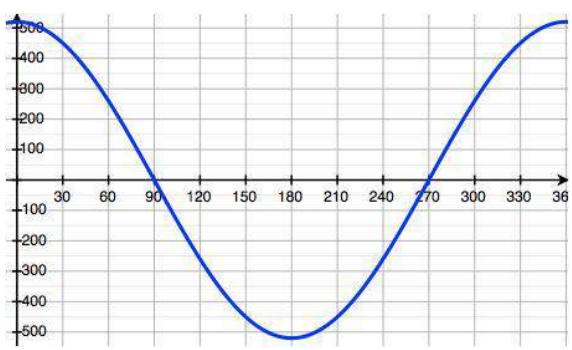
The radius of the wheel is the distance from the center of the wheel to a point on the wheel. We can easily measure this at one of the four points when the car is at the top or bottom of the wheel or at the far left or the far right. Thus, the radius is the difference between the maximum value of either function and zero, so the radius is the maximum value of either the height or the co-height function.

2. **In 2014, the High Roller Ferris wheel opened in Las Vegas, dwarfing the Seattle Great Wheel with a diameter of 520 feet. Sketch graphs of the height and co-height functions for one complete turn of the High Roller.**

Height:



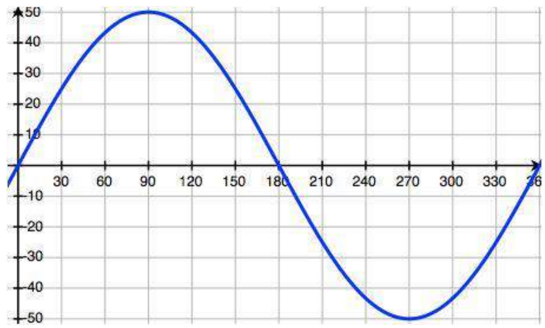
Co-Height:



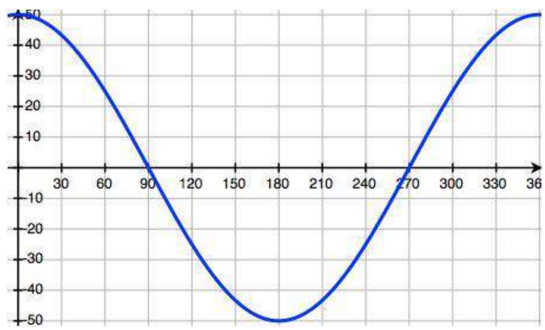
3. **Consider a Ferris wheel with a 50-foot radius. We will track the height and co-height of passenger cars that begin at the 3 o'clock position. Sketch graphs of the height and co-height functions for the following scenarios.**

   a. **A passenger car on the Ferris wheel completes one turn, traveling counterclockwise.**

   Height:

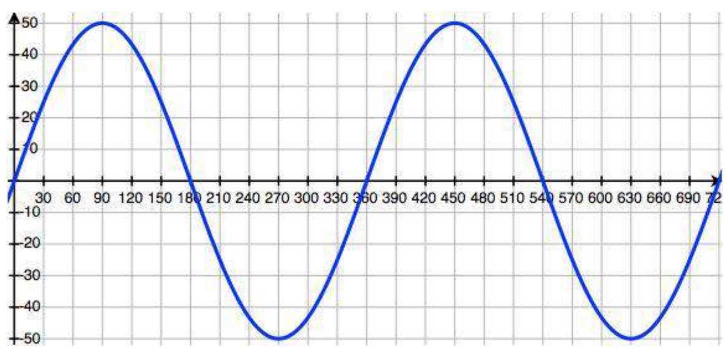


   Co-Height:

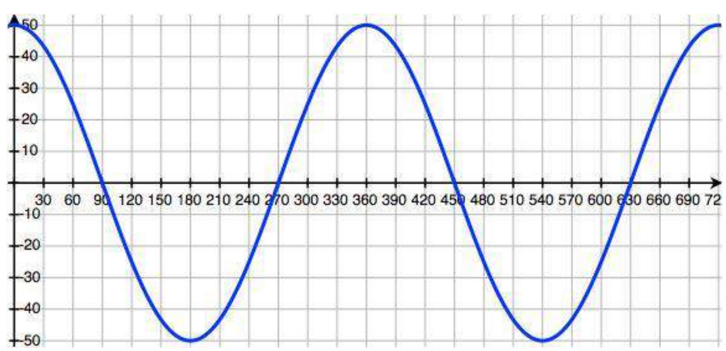


   b. **A passenger car on the Ferris wheel completes two full turns, traveling counterclockwise.**

   Height:

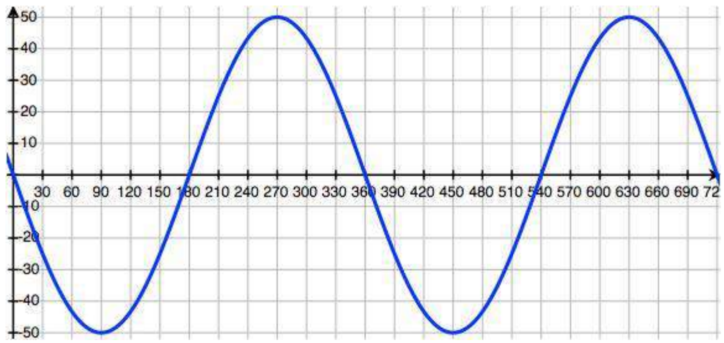


   Co-Height:

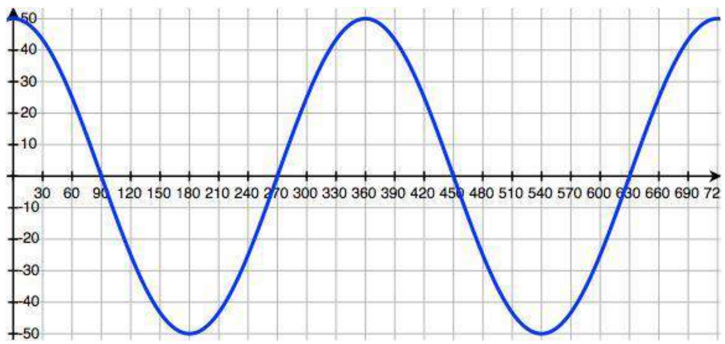


c. **The Ferris wheel is stuck in reverse, and a passenger car on the Ferris wheel completes two full *clockwise* turns.**

Height:



Co-Height:



4. **Consider a Ferris wheel with radius of 40 feet that is rotating counterclockwise. At which amounts of rotation are the values of the height and co-height functions equal? Does this result hold for a Ferris wheel with a different radius?**

Consider the right triangle formed by the spoke of the wheel connecting the car to the center, the horizontal axis, and the perpendicular line dropped from the car's position to the horizontal axis. If the value of the height and co-height functions are equal, then the legs of this triangle have the same length, meaning that it is an isosceles right triangle. There are four locations for such a triangle, with the passenger car being located in the first, second, third, or fourth quadrant. However, in the second and fourth quadrants, either the co-height takes on a negative value or the height takes on a negative value, but not both. Thus, for the co-height and height to take on the same value, the passenger car must be in either the first or the third quadrant. In the first quadrant, the car has rotated through $45°$, and in the third quadrant, the car has rotated through $180° + 45° = 225°$.

The same result holds for a Ferris wheel of any radius.

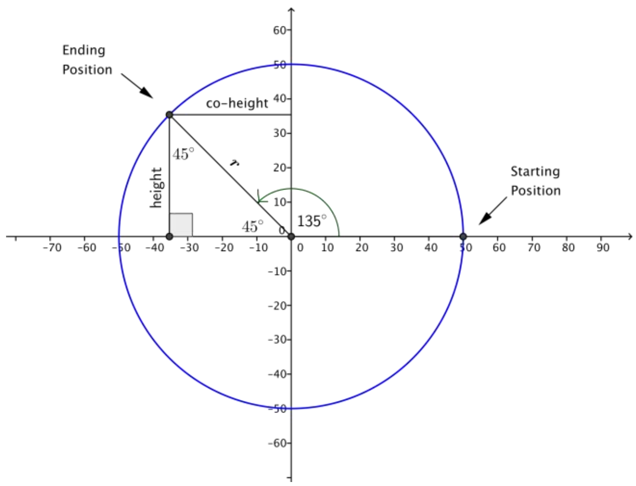
5. **Yuki is on a passenger car of a Ferris wheel at the 3 o'clock position. The wheel then rotates 135 degrees counterclockwise and gets stuck. Lee argues that she can compute the value of the co-height of Yuki's car if she is given one of the following two pieces of information:**

   i. **The value of the height function of Yuki's car, or**

   ii. **The diameter of the Ferris wheel itself.**

   Is Lee correct? Explain how you know.

   Lee is correct. Since Yuki's car started at the 3 o'clock position and rotated $135°$, then the ending position is in the second quadrant. The spoke of the Ferris wheel connecting her car to the center of the wheel makes a $45°$ angle with the horizontal, which creates a $45°$–$45°$–$90°$ triangle as shown in the diagram below. Then the height and the co-height at this position are equal, since the legs of an isosceles right triangle are congruent. Thus, if Lee knows the value of the height function of Yuki's car, then she knows the value of the co-height at this position.



   If Lee knows the diameter of the wheel, then she knows the radius, r, which is half of the diameter. Then she knows the length of a leg of an isosceles right triangle with hypotenuse of length r is $\frac{\sqrt{2}}{2}r$. Thus, if Lee knows the length of the diameter of the wheel, then she can calculate Yuki's co-height.

Lesson 3: The Motion of the Moon, Sun, and Stars—Motivating Mathematics

Student Outcomes

- Students explore the historical context of trigonometry as a motion of celestial bodies in a presumed circular arc.
- Students describe the position of an object along a line of sight in the context of circular motion.
- Students understand the naming of the quadrants and why counterclockwise motion is deemed the positive direction of rotation in mathematics.

Lesson Notes

This lesson provides us with another concrete example of observable periodic phenomena before we abstractly define the sine and cosine functions used to model circular motion—the observable path of the sun across the sky as seen from the earth. The historical roots of trigonometry lie in the attempts of astronomers to understand the motion of the stars and planets and to measure distances between celestial objects.

In this lesson, students investigate the historical development of sine tables from ancient India, from before the sine function had its name, and discover the work of Aryabhata I of India (pronounced *air-yah-bah-tah*), who lived from 476–550 C.E. Avoid using the terms *sine* and *cosine* as long as possible, delaying their introduction until the end of this lesson when the ancient measurements are related to the triangle trigonometry that students saw in high school Geometry, which leads into the formal definitions of these functions in the next lesson. Throughout this lesson, refer to the functions that became the sine and cosine functions using their original Sanskrit names *jya* and *kojya*, respectively. At the end of this lesson, explain how the name *jya* transformed into the modern name *sine*.

Consider having students do further research on the topics in this lesson, including the Babylonian astronomical diaries, Aryabhata I, *jya*, and early trigonometry. A few Wikipedia pages to use as a starting point for this research are listed at the end of the lesson.

Classwork

The lesson opens with a provocative question with the intent of uncovering what students already know about the motion of the sun and the planets. If just the system of Earth and the sun are considered, disregarding the other planets, then Earth and the sun both revolve around the center of mass of the system, called the *barycenter*. Because the mass of the sun, 1.99×10^{30} kg, is far greater than the mass of Earth, 5.97×10^{24} kg, the barycenter is very close to the center of the sun. Thus, scientists refer to the convention that Earth *goes around the sun*. However, the apparent motion of the sun in the sky is due to the daily rotation of Earth on its axis, not the motion of Earth orbiting the sun.

Opening (7 minutes)

Read the following prompts aloud and discuss as a class.

Opening

Why does it look like the sun moves across the sky?

The sun essentially stays still, but the earth rotates on its axis once every 24 hours; so, if an observer were standing in one place for 24 hours, it would appear that the sun becomes visible in the east, travels across the sky, and sets in the west.

Is the sun moving, or are you moving?

Technically, both are moving, but the sun moves imperceptibly while an observer on the face of the earth rotates away from the sun. I would say that the observer is moving.

In ancient Greek mythology, the god Helios was the personification of the sun. He rode across the sky every day in his chariot led by four horses. Why do your answers make it believable that in ancient times people imagined the sun was pulled across the sky each day?

If we did not know that the earth rotated every day, then a reasonable explanation would be that the sun moved across the sky each day.

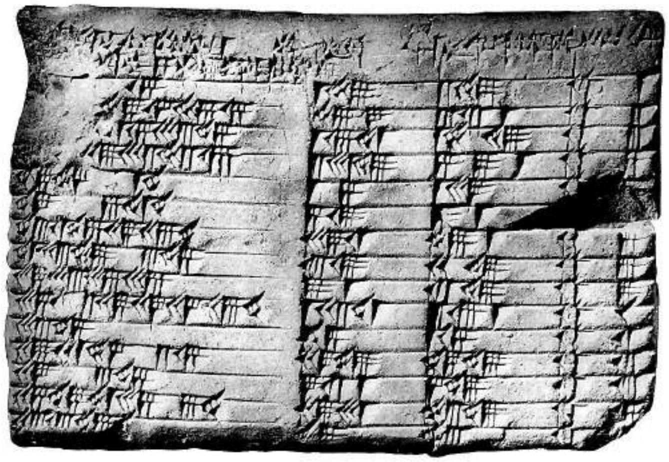
- Today, we know that Earth revolves around an axis once every 24 hours, and that rotation causes a period of sunlight and a period of darkness that we call *a day*.
- For this lesson, imagine that we are stationary and the sun is moving. In fact, we can imagine that we are at the center of a giant Ferris wheel and the sun is one of the passengers.

This section includes historical facts to lay the groundwork for the example from the beginnings of trigonometry. Although many ancient civilizations have left records of astronomical observations, including the Babylonians and ancient Greeks, the focus of the lesson is on some work done in ancient India—primarily the work of Aryabhata I, born in the year 476 C.E.

- Babylon was a city-state founded in 2286 B.C.E. in Mesopotamia, located about 85 miles south of Baghdad in modern day Iraq.
- The Babylonians wrote in cuneiform, a system of making marks on clay tablets with a stylus made from a reed with a triangular tip. Because clay is a fairly permanent medium, many tablets have survived until modern times, which means that we have a lot of information about how they approached mathematics and science. It wasn't until 1836 that the French scholar Eugene Burnouf began to decipher cuneiform so we could read these documents. For example, the tablet shown is known as Plimpton 322 and dates back to 1800 B.C.E. It contains a table of Pythagorean triples (a, b, c) written in cuneiform using the Babylonian base-60 number system.

Scaffolding:

The words *Babylon*, *cuneiform*, *stylus*, and *decipher* may need repeated choral rehearsal. The image can be used to illustrate and explain the word *cuneiform*.



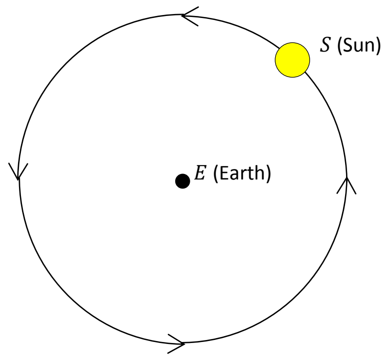
- Babylonian astronomers recorded astronomical phenomena in a set of tablets now known as The Babylonian Astronomical Diaries beginning around the year 750 B.C.E. and continuing through the first century B.C.E.
- The Babylonians observed that stars move in large and roughly circular arcs. From our perspective on Earth, the most prominent star, the sun, rises in the east, travels overhead, and then settles in the west prior to repeating its cycle in another twelve hours.
- Babylonian mathematics was influential on the mathematical development of the Egyptians and Greeks, and the Greeks traded goods and knowledge with ancient India. It was in ancient India that new mathematics was developed to describe these celestial observations. The work of the astronomer-mathematician Aryabhata I, born in 476 C.E., is of particular interest to our discussion.
- In ancient times, scholars assumed a *geocentric model* of the solar system, meaning that Earth was the center of the solar system, and all planets and stars and the sun revolved around Earth. We are assuming this model, even though it contradicts modern scientific convention. In 1532 C.E., Nicolas Copernicus proposed a heliocentric model of the solar system in which Earth and the other planets revolved around the sun. This model was very controversial at the time but has allowed modern scientists to understand the nature of our solar system and our universe.
- We assume the Babylonian conjecture—that stars and the sun travel on circular paths around the center of Earth. Then, we model a star's physical path relative to an observation point on Earth using the mathematics developed by Aryabhata I.

After discussing, ask students to summarize the information to a neighbor. Use this as a moment to informally assess understanding by listening to conversations.

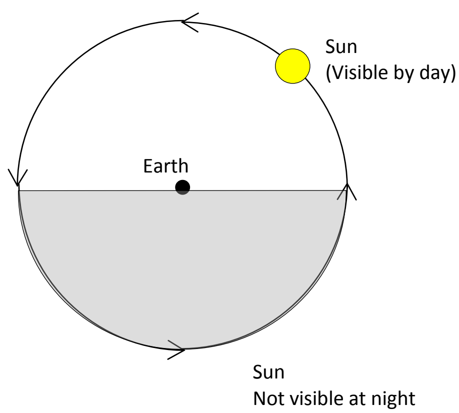
The information above is provided as background information to present to students. At a minimum, set the stage for Example 1 by announcing that the class is traveling back in time to the earliest uses of what is now called trigonometry. Ancient astronomers, who at the time believed that the sun and all other celestial bodies revolved around Earth, realized that they could model a star's physical path relative to an observation point on Earth using mathematics. Introduce the terms *geocentric* and *heliocentric* to students, and post the words on a word wall in the classroom. Also, discuss that while Greek and Babylonian mathematicians and astronomers tracked celestial bodies as they moved through the sky, this lesson focuses on the work of Aryabhata I (b. 476 C.E.) of India.

Discussion (7 minutes)

Using the historical context, students model the apparent motion of the sun with Earth as the center of its orbit. Guide the class through this example, using the board. Students should follow along in their notes, copying diagrams as necessary. Begin by drawing a large circle on the board, with its center (a point) clearly indicated:

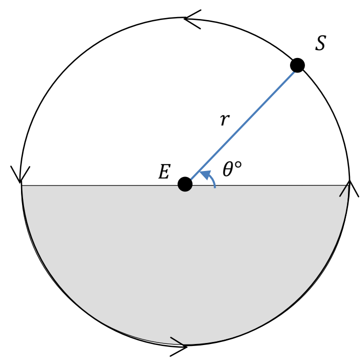


- In this model, we consider an observer at a fixed point E on Earth, and we condense the sun to a point S that moves in an apparent circular path around point E.
- Suppose that the observer faces north. To this person, the sun appears to travel counterclockwise around a semicircle, appearing (rising) in the east and disappearing (setting) in the west. Then, the path of the sun can be modeled by a circle with the observer at the center, and the portion where the sun can be seen is represented by the semicircle between the easternmost and westernmost points. Note to teacher: This setup does put the center of the circle as a point on the surface of the earth and not as the point at the center of the earth. We can disregard this apparent discrepancy because the radius of Earth, 3,963 miles, is negligible when compared to the 93 million-mile distance from Earth to the sun.
- A full revolution of the sun, S, around the circle represents a day, or 24 hours.
- To our observer, the sun is not visible at night. How can we reflect this fact in our model?
  - When the sun travels along the lower semicircle (i.e., below the line of the horizon), the sun is not visible to the observer. So we shade the lower half of the circle, which reflects nighttime. We do not consider the position of the sun when it cannot be seen.



- Recall that the sun rises in the east and sets in the west, so its apparent motion in our model is counterclockwise. Indeed, this is where our notion of clockwise and counterclockwise comes from; if we are in the Northern Hemisphere, as the sun moves counterclockwise, shadows move clockwise. Thus, the shade on a sundial used to mark the passage of time moves in the opposite direction as the sun across the sky. This is why our clocks go *clockwise*, but the sun travels *counterclockwise*.

- Accordingly, to model the movement of the sun, we measure the angle of elevation of the sun as it moves counterclockwise, as shown:



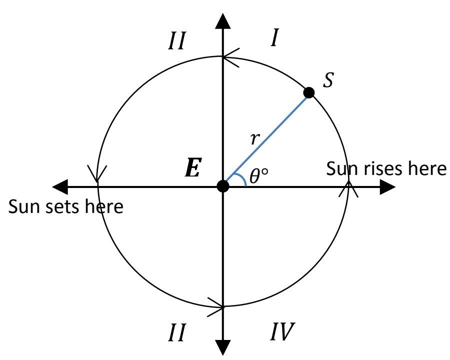
Discuss the information provided in the Student Materials.

Discussion

In mathematics, counterclockwise rotation is considered to be the positive direction of rotation, which runs counter to our experience with a very common example of rotation: the rotation of the hands on a clock.

- **Is there a connection between counterclockwise motion being considered to be positive and the naming of the quadrants on a standard coordinate system?**

  Yes! The quadrants are ordered according to the (counterclockwise) path the sun takes, beginning when it rises from the east.



- **What does the circle's radius, r, represent?**

  The radius length r represents the distance between the center of the earth and the center of the sun.

- We can describe the position of the sun relative to the earth using only the distance r and the angle of elevation of the sun with the horizon (the horizontal line through point E). Since we are assuming the sun travels along a circular path centered at E, the radius is constant. Thus, the location of the sun is determined solely by the angle of elevation θ measured in degrees.
- Remember that astronomers were interested in the *height* of the stars and planets above Earth. Since it is not possible to just build a very tall ladder and take measurements, they had to devise other ways to estimate these distances. This notion of measuring the *heig*ht of the stars and planets above Earth is similar to how we measured the height of the passenger car on a Ferris wheel in the previous lessons.

Before introducing the work of Aryabhata that led to trigonometry, have students summarize the work in this example by responding to the question and discussing their thoughts with a partner. Have a few students share out to the whole class as well.

- **How has the motion of the sun influenced the development of mathematics?**

  The naming of the quadrants and the idea of measuring rotations counterclockwise make sense in terms of how we perceive the movement of stars and planets on Earth.

- **How is measuring the *height* of the sun like measuring the Ferris wheel passenger car height in the previous lessons?**

  Both situations relate a vertical distance to a measurement of rotation from an initial reference point.

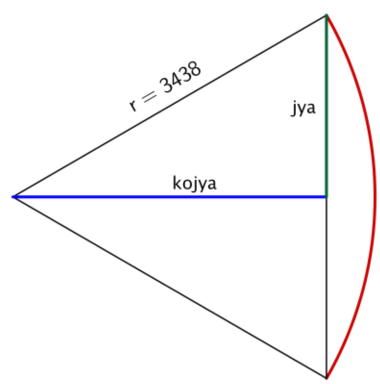
Discussion (4 minutes)

In this section, we discuss the historical terms used by Aryabhata I in his astronomical work that led to trigonometry. Sanskrit names are spelled phonetically in the Roman alphabet, so Aryabhata is pronounced *Air-yah-bah-ta*.

- The best-known work of the Indian scholar Aryabhata I, born in 476 C.E., is the *Aryabhatiya*, a compilation of mathematical and astronomical results that was not only used to support new developments in mathematics in India and Greece but also provides us with a snapshot of what was known at that time. We are using his developments to model the position of the sun in the sky. Aryabhata I spoke and wrote in the Sanskrit language.
- An arc of a circle together with the chord joining the ends of the arc are shaped like a bow and bowstring. The Sanskrit word for bowstring is *samastajya*. Indian mathematicians often used the half-chord *ardhajya,* so Aryabhata I used the abbreviation *jya* (pronounced *jhah*) for half of the length of the chord joining the endpoints of the arc of the circle.
- The term *koti-jya*, often abbreviated in Indian texts as *kojya*, means the side of a right-angled triangle, one of whose sides is the jya.
- Indian astronomers used a value of $r = 3438$ to represent the distance from the earth to the sun because this is roughly the radius of a circle whose circumference is $360°$, which is $(360 \cdot 60')$ or $21{,}600'$, measured in minutes.

Scaffolding:

- The words *Aryabhata*, *jya*, and *kojya* can be chorally repeated by students. Point to these terms on the image while asking students to repeat them out loud.
- For advanced learners, describe the terms in words only and have them draw the diagram or label the appropriate parts of a diagram without having seen the image first.



Exercises 1–3 (12 minutes)

In his text *Aryabhatiya*, Aryabhata constructed a table of values of the jya in increments of $3\frac{3}{4}$ degrees, which were used to calculate the positions of astronomical objects. The recursive formula he used to construct this table is as follows:

$$s_1 = 225$$
$$s_{n+1} = s_n + s_1 - \frac{(s_1 + s_2 + \cdots + s_n)}{s_1}$$

where n counts the increments of $3\frac{3}{4}°$. Then, $\text{jya}(n \cdot 3°) = s_n$.

Exercises 1–4

1. Calculate $\text{jya}(7°)$, $\text{jya}(11°)$, $\text{jya}(15°)$, and $\text{jya}(18°)$ using Aryabhata's formula[1], round to the nearest integer, and add your results to the table below. Leave the rightmost column blank for now.

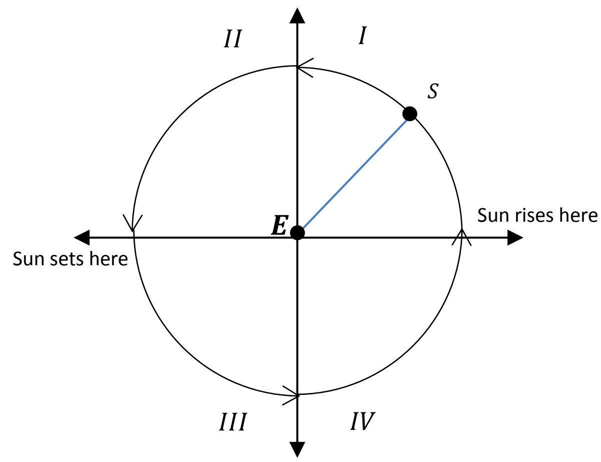
n	θ, in degrees	$\text{jya}(\theta°)$	$3438\sin(\theta°)$
1	$3\frac{3}{4}$	225	
2	$7\frac{1}{2}$	449	
3	$11\frac{1}{4}$	671	
4	15	890	
5	$18\frac{3}{4}$	1105	
6	$22\frac{1}{2}$	1315	
7	$26\frac{1}{4}$	1520	
8	30	1719	
9	$33\frac{3}{4}$	1910	
10	$37\frac{1}{2}$	2093	
11	$41\frac{1}{4}$	2267	
12	45	2431	

n	θ, in degrees	$\text{jya}(\theta°)$	$3438\sin(\theta°)$
13	$48\frac{3}{4}$	2585	
14	$52\frac{1}{2}$	2728	
15	$56\frac{1}{4}$	2859	
16	60	2978	
17	$63\frac{3}{4}$	3084	
18	$67\frac{1}{2}$	3177	
19	$71\frac{1}{4}$	3256	
20	75	3321	
21	$78\frac{3}{4}$	3372	
22	$82\frac{1}{2}$	3409	
23	$86\frac{1}{4}$	3431	
24	90	3438	

[1]In constructing the table, Aryabhata made adjustments to the values of his approximation to the jya to match his observational data. The first adjustment occurs in the calculation of $\text{jya}(30°)$. Thus, the entire table cannot be accurately constructed using this formula.

MP.7

2. **Label the angle θ, $\text{jya}(\theta°)$, $\text{kojya}(\theta°)$, and r in the diagram shown below.**



a. **How does this relate to something you've done before?**

Jya and kojya are lengths of the sides of a right triangle. They are also like the height and co-height functions of the passenger car on a Ferris wheel.

b. **How do $\text{jya}(\theta°)$ and $\text{kojya}(\theta°)$ relate to lengths *we* already know?**

From prior work with triangle trigonometry, $\text{jya}(\theta°) = r\sin(\theta°)$ and $\text{kojya}(\theta°) = r\cos(\theta°)$.

3. **Use your calculator to compute $r\sin(\theta°)$ for each value of θ in the table from Exercise 1, where $r = 3438$. Record this in the blank column on the right in Exercise 1, rounding to the nearest integer. How do Aryabhata's approximated values from around the year 500 C.E. compare to the value we can calculate with our modern technology?**

The values are surprisingly close.

Discussion (3 minutes)

If we set $r = 1$, then $\text{jya}(\alpha°) = \sin(\alpha°)$, and $\text{kojya}(\alpha°) = \cos(\alpha°)$; so, Aryabhata constructed the first known sine table in mathematics. This table was used to calculate the positions of the planets, the stars, and the sun in the sky.

If the first instance of the function we know now as the sine function started off with the Sanskrit name *jya*, how did it get to be called *sine*?

- We know that Aryabhata referred to this length as *jya*.
- Transcribed letter-by-letter into Arabic in the 10th century, *jya* became *jiba*. In medieval writing, scribes regularly omitted vowels to save time, space, and resources, so the Arabic scribes wrote just *jb*.
- Since *jiba* isn't a real word in Arabic, later readers interpreted *jb* as *jaib*, which is an Arabic word meaning *cove* or *bay*.
- When translated into Latin around 1150 C.E., *jaib* became *sinus*, which is the Latin word for *bay*.
- *Sinus* got shortened into *sine* in English.

Exercise 4 (7 minutes)

Have students complete this exercise in groups. While part (a) is relatively straightforward, part (b) requires some thought and discussion.

> *Scaffolding:*
> Encourage struggling students to draw a semicircle to represent the path of the sun across the sky. Label the intersections with the horizontal and vertical axes as 6:00 a.m., 12:00 p.m., and 6:00 p.m.

4. We will assume that the sun rises at 6:00 a.m., is directly overhead at 12:00 noon, and sets at 6:00 p.m. We measure the *height* of the sun by finding its vertical distance from the horizon line; the horizontal line that connects the easternmost point, where the sun rises, to the westernmost point, where the sun sets.

a. Using $r = 3438$, as Aryabhata did, find the *height* of the sun at the times listed in the following table:

Time of day	Height
6:00 a.m.	
7:00 a.m.	
8:00 a.m.	
9:00 a.m.	
10:00 a.m.	
11:00 a.m.	
12:00 p.m.	

Given this scenario, the sun will move 15° each hour. Then the heights will be:

6:00 a.m. $\rightarrow \text{jya}(0^\circ) = 0$

7:00 a.m. $\rightarrow \text{jya}(15^\circ) = 890$

8:00 a.m. $\rightarrow \text{jya}(30^\circ) = 1719$

9:00 a.m. $\rightarrow \text{jya}(45^\circ) = 2431$

10:00 a.m. $\rightarrow \text{jya}(60^\circ) = 2978$

11:00 a.m. $\rightarrow \text{jya}(75^\circ) = 3321$

12:00 p.m. $\rightarrow \text{jya}(90^\circ) = 3438$

b. Now, find the height of the sun at the times listed in the following table using the actual distance from the earth to the sun, which is 93 million miles.

Time of day	Height
6:00 a.m.	
7:00 a.m.	
8:00 a.m.	
9:00 a.m.	
10:00 a.m.	
11:00 a.m.	
12:00 p.m.	

The sun will move 15° each hour, and we calculate the height at position θ by $\frac{\text{jya}(\theta^\circ)}{r} \cdot 93$. Our units are millions of miles. Then, the heights, in millions of miles, will be:

6:00 a.m. $\rightarrow 93 \cdot \frac{\text{jya}(0^\circ)}{3438} = 0.0$

7:00 a.m. $\rightarrow 93 \cdot \frac{\text{jya}(15^\circ)}{3438} = 24.1$

8:00 a.m. $\rightarrow 93 \cdot \frac{\text{jya}(30^\circ)}{3438} = 46.5$

9:00 a.m. $\rightarrow 93 \cdot \frac{\text{jya}(45^\circ)}{3438} = 65.8$

10:00 a.m. $\rightarrow 93 \cdot \frac{\text{jya}(60^\circ)}{3438} = 80.6$

11:00 a.m. $\rightarrow 93 \cdot \frac{\text{jya}(75^\circ)}{3438} = 89.8$

12:00 p.m. $\rightarrow 93 \cdot \frac{\text{jya}(90^\circ)}{3438} = 93$

Closing (2 minutes)

Have students respond to the following questions, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

- Our exploration of the historical development of the sine table is based on observations of the motion of the planets and stars in Babylon and India. Is it based on a geocentric or heliocentric model? What does that term mean?
  - *It is based on a geocentric model, which assumes that celestial bodies rotate around Earth.*
- How is Aryabhata's function jya related to the sine of an angle of a triangle?
  - *When the rotation is an acute angle, the function* jya *is the same as the sine of an angle of a triangle.*
- How does the apparent motion of the sun in the sky relate to the motion of a passenger car of a Ferris wheel?
  - *Using an ancient (and outdated) geocentric model of the solar system, we can think of the sun as a passenger car on a gigantic Ferris wheel, with the radius the distance from the earth to the sun.*

The following facts are some important summary elements.

> Lesson Summary
>
> **Ancient scholars in Babylon and India conjectured that celestial motion was circular; the sun and other stars orbited the earth in a circular fashion. The earth was presumed to be the center of the sun's orbit.**
>
> **The quadrant numbering in a coordinate system is consistent with the counterclockwise motion of the sun, which rises from the east and sets in the west.**
>
> **The 6th century Indian scholar Aryabhata created the first sine table, using a measurement he called *jya*. The purpose of his table was to calculate the position of the sun, the stars, and the planets.**

Exit Ticket (3 minutes)

References:

http://en.wikipedia.org/wiki/Jya

http://en.wikipedia.org/wiki/Aryabhata

http://en.wikipedia.org/wiki/Babylonian_astronomical_diaries

T. Hayashi, "Aryabhata's Rule and Table for Sine-Differences", *Historia Mathematica* 24 (1997), 396–406.

The Crest of the Peacock, 3rd Edition, George Gheverghese Joseph, Princeton University Press, 2011.

An Introduction to the History of Mathematics, 6th Edition, Howard Eves, Brooks-Cole, 1990.

A History of Mathematics, 2nd Edition, Carl B. Boyer, Wiley & Sons, 1991.

Name ____ Date ____________________

Lesson 3: The Motion of the Moon, Sun, and Stars—Motivating Mathematics

Exit Ticket

1. Explain why counterclockwise is considered to be the positive direction of rotation in mathematics.

2. Suppose that you measure the angle of elevation of your line of sight with the sun to be $67.5°$. If we use the value of 1 astronomical unit (abbreviated AU) as the distance from the earth to the sun, use the portion of the jya table below to calculate the sun's apparent height in astronomical units.

$\boldsymbol{\theta}$, in degrees	jya$(\boldsymbol{\theta}°)$
$48\frac{3}{4}$	2585
$52\frac{1}{2}$	2728
$56\frac{1}{4}$	2859
60	2978
$63\frac{3}{4}$	3084
$67\frac{1}{2}$	3177
$71\frac{1}{4}$	3256

Exit Ticket Sample Solutions

1. **Explain why counterclockwise is considered to be the positive direction of rotation in mathematics.**

   Clocks were invented based on the movement of the shadows across a sundial. These shadows then move in the direction we call clockwise. The sun, on the other hand, moves in the opposite direction as the shadows, so the sun appears to move counterclockwise with respect to us if we are facing north in the Northern Hemisphere. Since our observations are based on the movement of the sun, the direction of the sun's path, starting in the east, rising, and setting in the west, determined our conventions for the direction of rotation considered positive in the coordinate plane.

2. **Suppose that you measure the angle of elevation of your line of sight with the sun to be $67.5°$. If we use the value of 1 astronomical unit (abbreviated AU) as the distance from the earth to the sun, use the portion of the jya table below to calculate the sun's apparent height in astronomical units.**

   The only value we need from the table is $\text{jya}(67.5°) = 3177$. *Since we are using a radius of* 1 *astronomical unit, the apparent height will be smaller than* 1. *We have:*

   $$\left(\frac{\text{jya}(67.5°)}{3438}\right) \cdot 1 \text{ AU} = \frac{3177}{3438}\text{AU} = 0.924 \text{ AU}.$$

θ, in degrees	$\text{jya}(\theta°)$
$48\frac{3}{4}$	2585
$52\frac{1}{2}$	2728
$56\frac{1}{4}$	2859
60	2978
$63\frac{3}{4}$	3084
$67\frac{1}{2}$	3177
$71\frac{1}{4}$	3256

Problem Set Sample Solutions

1. **An Indian astronomer noted that the angle of his line of sight to Venus measured $52°$. We now know that the average distance from Earth to Venus is 162 million miles. Use Aryabhata's table to estimate the apparent height of Venus. Round your answer to the nearest million miles.**

   By the table, $\text{jya}(52.5°) = 2728$. *Since* $162 \cdot \frac{\text{jya}(52.5°)}{3438} = 129$, *the apparent height is* 129 *million miles.*

2. **Later, the Indian astronomer saw that the angle of his line of sight to Mars measured $82°$. We now know that the average distance from Earth to Mars is 140 million miles. Use Aryabhata's table to estimate the apparent height of Mars. Round your answer to the nearest million miles.**

   By the table, $\text{jya}(82.5°) = 3409$. *Since* $140 \cdot \frac{\text{jya}(82.5°)}{3438} = 139$, *the apparent height is* 139 *million miles.*

3. The moon orbits the earth in an elongated orbit, with an average distance of the moon from the earth of roughly $239,000$ miles. It takes the moon 27.32 days to travel around the earth, so the moon moves with respect to the stars roughly $0.5°$ every hour. Suppose that angle of inclination of the moon with respect to the observer measures $45°$ at midnight. As in Example 1, an observer is standing still and facing north. Use Aryabhata's jya table to find the apparent height of the moon above the observer at the times listed in the table below, to the nearest thousand miles.

Time (hour:min)	Angle of elevation θ, in degrees	Height
12:00 a.m.		
7:30 a.m.		
3:00 p.m.		
10:30 p.m.		
6:00 a.m.		
1:30 p.m.		
9:00 p.m.		

Students must realize that every 7.5 hours, the moon travels $0.5° \times 7.5 = 3\frac{3}{4}°$. Thus, we approximate the apparent height of the moon by $\frac{\text{jya}(\theta°)}{3438} \cdot 239,000$ when the angle of elevation is θ.

12:00 a.m. → $\theta = 45$; $239,000 \cdot \frac{\text{jya}(45°)}{3438} \approx 169,000$ *Apparent height is $169,000$ miles.*

7:30 a.m. → $\theta = 48\frac{3}{4}$; $239,000 \cdot \frac{\text{jya}\left(48\frac{3}{4}°\right)}{3438} \approx 180,000$ *Apparent height is $180,000$ miles.*

3:00 p.m. → $\theta = 52\frac{1}{2}$; $239,000 \cdot \frac{\text{jya}\left(52\frac{1}{2}°\right)}{3438} \approx 190,000$ *Apparent height is $190,000$ miles.*

10:30 p.m. → $\theta = 56\frac{1}{4}$; $239,000 \cdot \frac{\text{jya}\left(56\frac{1}{4}°\right)}{3438} \approx 199,000$ *Apparent height is $199,000$ miles.*

6:00 a.m. → $\theta = 60$; $239,000 \cdot \frac{\text{jya}(60°)}{3438} \approx 207,000$ *Apparent height is $207,000$ miles.*

1:30 p.m. → $\theta = 63\frac{3}{4}$; $239,000 \cdot \frac{\text{jya}\left(63\frac{3}{4}°\right)}{3438} \approx 214,000$ *Apparent height is $214,000$ miles.*

9:00 p.m. → $\theta = 67\frac{1}{2}$; $239,000 \cdot \frac{\text{jya}\left(67\frac{1}{2}°\right)}{3438} \approx 221,000$ *Apparent height is $221,000$ miles.*

4. George wants to apply Aryabhata's method to estimate the height of the International Space Station, which orbits Earth at a speed of about $17,500$ miles per hour. This means that the space station makes one full rotation around Earth roughly every 90 minutes. The space station maintains a low earth orbit, with an average distance from Earth of 238 miles.

   a. George supposes that the space station is just visible on the eastern horizon at 12:00 midnight, so its apparent height at that time would be 0 miles above the horizon. Use Aryabhata's jya table to find the apparent height of the space station above the observer at the times listed in the table below.

Time (hour:min:sec)	Angle of elevation θ, in degrees	Height
12:00:00 a.m.		
12:03:45 a.m.		
12:07:30 a.m.		
12:11:15 a.m.		
12:15:00 a.m.		
12:18:45 a.m.		
12:22:30 a.m.		

Students must realize that every 90 minutes the space station travels one full revolution, so in 3 minutes, the space station travels through a rotation of $\frac{1}{90}\cdot\frac{15}{4}(360°) = 15°$. Thus, we use $\frac{\text{jya}(n\cdot 15°)}{3438}\cdot 238$ for the height at $n\cdot 3$ minutes after 12:00 a.m.

12:00:00 a.m. $\rightarrow \theta = 0$; $238\cdot\frac{\text{jya}(0°)}{3438} = 0$ *The apparent height is 0 miles.*

12:03:45 a.m. $\rightarrow \theta = 15$; $238\cdot\frac{\text{jya}(15°)}{3438} = 62$ *The apparent height is 62 miles.*

12:07:30 a.m. $\rightarrow \theta = 30$; $238\cdot\frac{\text{jya}(30°)}{3438} = 119$ *The apparent height is 119 miles.*

12:11:15 a.m. $\rightarrow \theta = 45$; $238\cdot\frac{\text{jya}(45°)}{3438} = 168$ *The apparent height is 168 miles.*

12:15:00 a.m. $\rightarrow \theta = 60$; $238\cdot\frac{\text{jya}(60°)}{3438} = 206$ *The apparent height is 206 miles.*

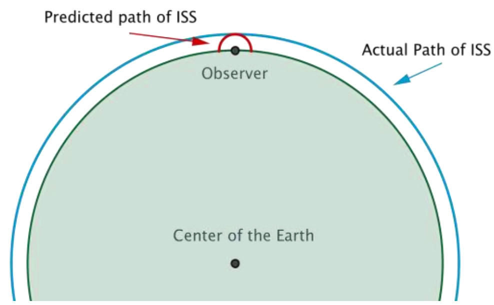
12:18:45 a.m. $\rightarrow \theta = 75$; $238\cdot\frac{\text{jya}(75°)}{3438} = 230$ *The apparent height is 230 miles.*

12:22:30 a.m. $\rightarrow \theta = 90$; $238\cdot\frac{\text{jya}(90°)}{3438} = 238$ *The apparent height is 238 miles.*

MP.4

b. **When George presents his solution to his classmate Jane, she tells him that his model isn't appropriate for this situation. Is she correct? Explain how you know. (Hint: As we set up our model in the first discussion, we treated our observer as if he was the center of the orbit of the sun around the earth. In part (a) of this problem, we treated our observer as if she were the center of the orbit of the International Space Station around Earth. The radius of Earth is approximately $3,963$ miles, the space station orbits about 238 miles above Earth's surface, and the distance from Earth to the sun is roughly $93,000,000$ miles. Draw a picture of the earth and the path of the space station, and then compare that to the points with heights and rotation angles from part (a).)**

The semicircular path of the space station in this model will have a radius of 238 miles. However, because $3963 + 238 = 4201$, the space station should have a radius of $4,201$ miles. In this model, the space station starts at a point 238 miles to the east of the observer and crashes into Earth at a point 238 miles to the west of the observer. Thus, this is not an appropriate model to use for the height of the International Space Station.



The problem is that the radius of Earth is negligible in comparison to the distance of $93,000,000$ miles from the surface of Earth to the sun, but the radius of Earth is not negligible in comparison to the distance of 238 miles from the surface of Earth to the International Space Station.

Lesson 4: From Circle-ometry to Trigonometry

Student Outcomes

- Students define sine and cosine as functions for degrees of rotation of the ray formed by the positive x-axis up to one full turn.
- Students use special triangles to geometrically determine the values of sine and cosine for 30, 45, 60, and 90 degrees.

Lesson Notes

In the preceding lessons, students have developed the height and co-height functions of a passenger car on a Ferris wheel and considered the historical roots of trigonometry through developments in astronomy. From these experiences, we extract the meaning of the sine and cosine of a number of degrees of rotation. For consistency with their past experiences with triangle trigonometry, we need to demonstrate that our new functions of sine and cosine are generalizations of the sine and cosine functions of an angle in a triangle studied in geometry. For this lesson confine discussion to rotations by a number of degrees between 0 and 360. Lesson 5 extends the domain of the sine and cosine functions to the entire real line, and Lesson 9 transitions from measuring rotation in degrees to measuring rotation in radians. This entire lesson should be taught without using calculators.

Notating Trigonometric Functions: It is convenient, as adults, to use the notation $\sin^2 x$ to refer to the value of the square of the sine function. However, rushing too fast to this abbreviated notation for trigonometric functions leads to incorrect understandings of how functions are manipulated, which can lead students to think that $\sin x$ is short for $\sin \cdot x$ and incorrectly divide out the variable, so that $\frac{\sin x}{x}$ becomes $\sin$.

To reduce these types of common notation-driven errors later, this curriculum is very deliberate about how and when we use abbreviated function notation for sine, cosine, and tangent:

1. In Geometry, sine, cosine, and tangent are thought of as the value of ratios of triangles, not as functions. No attempt is made to describe the trigonometric ratios as functions on the real line. Therefore, the notation is just an abbreviation for the *sine of an angle* ($\sin \angle A$) or *sine of an angle measure* ($\sin \theta$). Parentheses are used more for grouping and clarity reasons than as symbols used to represent a function.
2. In Algebra II, to distinguish between the ratio version of sine in geometry, all sine functions are notated as functions: $\sin(x)$ is the value of the sine function for the real number x, just like $f(x)$ is the value of the function f for the real number x. In this course, strictly maintain parentheses as part of function notation, writing for example, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ instead of $\tan \theta = \frac{\sin \theta}{\cos \theta}$, to maintain function notation integrity. The expression $(\sin(\theta))^2$ is abbreviated by $\sin^2(\theta)$ in Algebra II, maintaining the use of function notation through the use of parentheses.
3. By Precalculus and Advanced Topics, students have had two full years of working with sine, cosine, and tangent as both ratios and functions. It is finally in this year that we begin to blur the distinction between ratio and function notations and write, for example, $\sin^2 \theta$ as the value of the square of the sine function for the real number θ, which is how most calculus textbooks notate these functions.

Classwork

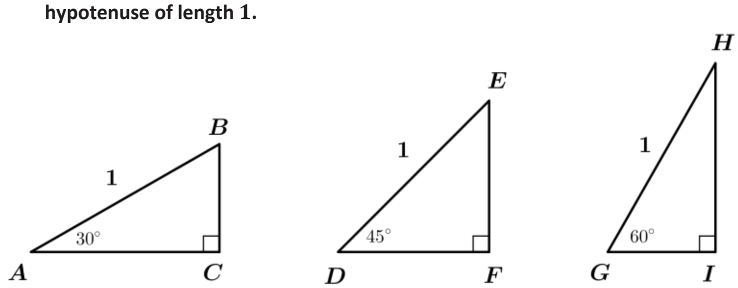
We begin the lesson with an opening exercise that requires that students find the sine and cosine of an angle in a right triangle with given side lengths so that they recall the previous definitions of the trigonometric ratios. We need students to recall the side lengths of the *special triangles* from Geometry, so that is also part of the Opening Exercise.

Opening Exercises (4 minutes)

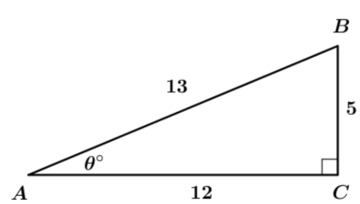
Allow students to work in pairs or small groups to encourage recall of triangle trigonometry from Geometry. Do not allow the use of calculators.

Opening Exercises

1. **Find the lengths of the sides of the right triangles below, each of which has hypotenuse of length 1.**

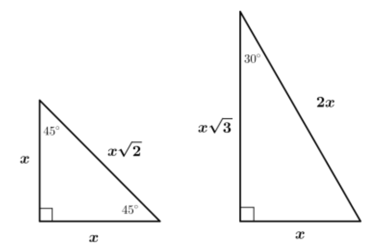


2. **Given the following right triangle $\triangle ABC$ with $m\angle A = \theta°$, find $\sin(\theta°)$ and $\cos(\theta°)$.**



Scaffolding:

- Place a chart at the front of the room showing the relationships between the special triangles (example shown below). Additionally, a visual of the definitions of $\sin(\theta°)$ and $\cos(\theta°)$ in terms of right triangles helps as well.



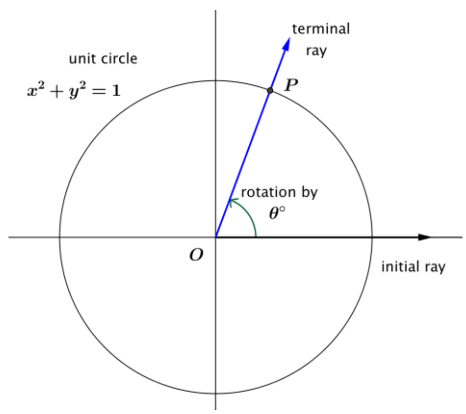
- For students who may be working above grade level, show a diagram with a $52°$ angle and hypotenuse 1. Ask them to hypothesize about the side lengths and justify their reasoning.

Discussion (6 minutes)

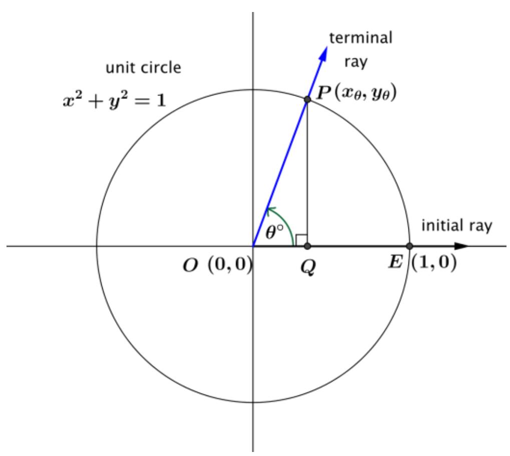
In Lessons 1 and 2 of this module, we defined the height and co-height functions for a passenger car travelling around a Ferris wheel. The following discussion builds students' abilities to employ MP.4 as they develop a function to model the real-world behavior of the Ferris wheel.

- What was the independent variable for these functions?
  - *The variable was the degrees of rotation of the Ferris wheel from the horizontal reference position to its current position.*
- Since cars on a Ferris wheel travel in a giant circle, can we just generalize height and co-height for movement around any circle? How could we do that?
  - *We can measure the vertical distance from the current point to the horizontal axis as we do on a Ferris wheel for the height function and measure the horizontal distance from the current point to the vertical axis for the co-height function.*

- The radius of the circle doesn't matter for our discussion since we are concerned with the degrees of rotation of the car on the wheel. So, for simplicity we just count *one radius length* as our unit, and then we are working on a circle with radius 1 unit. So, we suppose that our circle has radius 1 unit, and we put the circle on a coordinate grid. The simplest place to put the circle is centered at the origin. What is the equation of this circle?
    - *The equation of the circle is* $x^2 + y^2 = 1$.
- The circle with equation $x^2 + y^2 = 1$ is known as the *unit circle* because its radius is one unit.
- Just as the sun rises in the east and has an angle of elevation of 0 degrees at its easternmost point, we consider the point furthest to the right to be our point of reference. What are the coordinates of this point on the unit circle? How does this point relate to the Ferris wheel example from Lessons 1 and 2?



    - *The point* $(1,0)$ *is the point at the 3 o'clock position where our riders often boarded the Ferris wheel.*
- Consider the rotation of the *initial ray*, which is the ray formed by the positive x-axis, and let point P be the intersection of the initial ray with the unit circle. Suppose that the initial ray has been rotated θ degrees counterclockwise around the unit circle, where $0 < \theta < 90$, so that point P stays in the first quadrant.
- After the rotation of the initial ray by θ degrees, let the coordinates of point P be (x_θ, y_θ). Let O denote the center $(0,0)$ of the circle, and let E denote the reference point $(1,0)$. Drop a perpendicular segment from P to ray $\overrightarrow{OE}$ that intersects at point Q. What are the coordinates of point Q?



> *Scaffolding:*
>
> For students not quite ready for this level of abstraction, use the specific point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ instead of the generic point (x, y) to make the connection between height/co-height and sine/cosine. Then skip Example 1.

    - *The coordinates of* Q *are* $(x_\theta, 0)$.

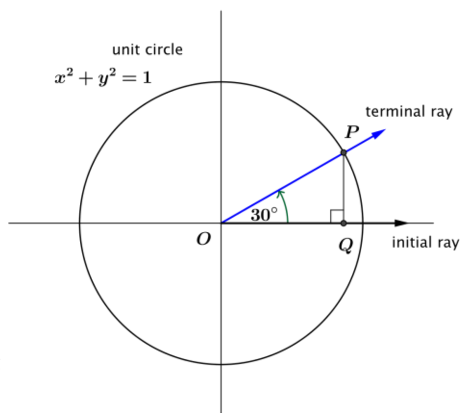
MP.4

- What do we know about the lengths OP, OQ, and QP?
  - *We know that $OP = 1$ because this is a circle of radius 1. Also, $OQ = x_{\theta}$, and $QP = y_{\theta}$.*
- What are the height and co-height of point P?
  - *The height of P is y_{θ}, and the co-height of P is x_{θ}.*
- What kind of triangle is $\triangle OQP$?
  - *A right triangle with right angle at Q*
- Using triangle trigonometry, what are $\sin(\theta°)$ and $\cos(\theta°)$?
  - *By trigonometry, $\sin(\theta°) = \frac{y_{\theta}}{1} = y_{\theta}$, and $\cos(\theta°) = \frac{x_{\theta}}{1} = x_{\theta}$.*
- What can we conclude about the height and co-height of point P and the sine and cosine of θ where $0 < \theta < 90$? In this case, the corresponding point P is in the first quadrant.
  - *If $0 < \theta < 90$, then $\sin(\theta°)$ is the same as the height of the corresponding point P, and $\cos(\theta°)$ is the same as the co-height of P.*

Example 1 (3 minutes)

Example 1

Suppose that point P is the point on the unit circle obtained by rotating the initial ray through $30°$. Find $\sin(30°)$ and $\cos(30°)$.



What is the length OQ of the horizontal leg of our triangle?

By remembering the special triangles from geometry, we have $OQ = \frac{\sqrt{3}}{2}$.

What is the length QP of the vertical leg of our triangle?

Either by the Pythagorean theorem or by remembering the special triangles from Geometry, we have $QP = \frac{1}{2}$.

What is $\sin(30°)$?

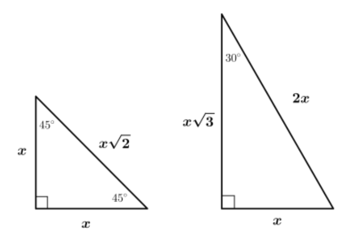
$\sin(30°) = \frac{1}{2}$

What is $\cos(30°)$?

$\cos(30°) = \frac{\sqrt{3}}{2}$

Scaffolding:

For struggling students, provide a review of the side lengths of 30°-60°-90° and 45°-45°-90° triangles.



Exercises 1–2 (4 minutes)

These exercises serve to review the special triangles from Geometry and to tie together the ideas of the height and co-height functions and the sine and cosine functions. Have students complete these exercises in pairs.

Exercises 1–2

1. **Suppose that P is the point on the unit circle obtained by rotating the initial ray through $45°$. Find $\sin(45°)$ and $\cos(45°)$.**

   We have $\sin(45°) = \frac{\sqrt{2}}{2}$ *and* $\cos(45°) = \frac{\sqrt{2}}{2}$.

2. **Suppose that P is the point on the unit circle obtained by rotating the initial ray through $60°$. Find $\sin(60°)$ and $\cos(60°)$.**

   We have $\sin(60°) = \frac{\sqrt{3}}{2}$ *and* $\cos(60°) = \frac{1}{2}$.

Discussion (3 minutes)

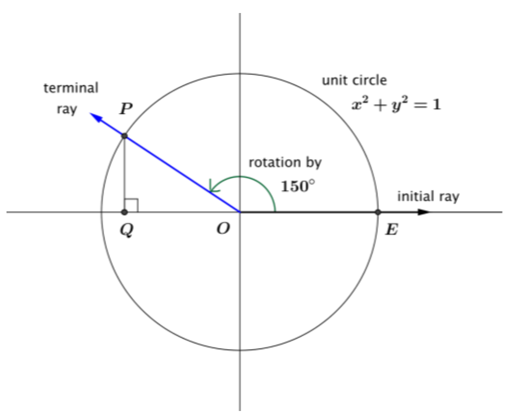
- Remember that sine and cosine are functions of the number of degrees of rotation of the initial horizontal ray moving counterclockwise about the origin. So far, we have only made sense of sine and cosine for degrees of rotation between 0 and 90, but the Ferris wheel doesn't just rotate $90°$ and then stop; it continues going around the full circle. How can we extend our ideas about sine and cosine to any counterclockwise rotation up to $360°$?
  - *Solicit ideas from the class. Guide them to realize that since the height and co-height functions are defined on all points of the circle, we can define sine and cosine for any number of degrees of rotation around the circle.*

Example 2 (3 minutes)

For this example, consider asking students to develop conjectures for $\sin(150°)$ and $\cos(150°)$ and to justify these conjectures with words or diagrams. This is an opportunity to build students' abilities with MP.3.

Example 2

Suppose that P is the point on the unit circle obtained by rotating the initial ray through $150°$. Find $\sin(150°)$ and $\cos(150°)$.



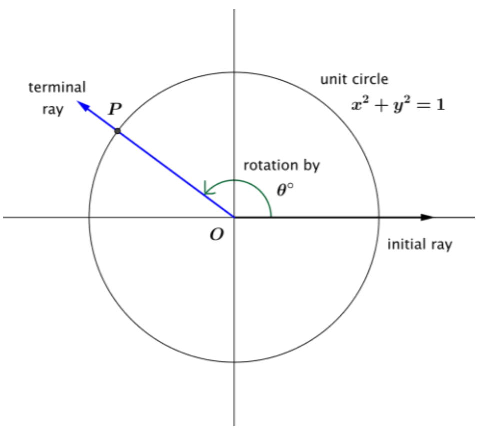
MP.3

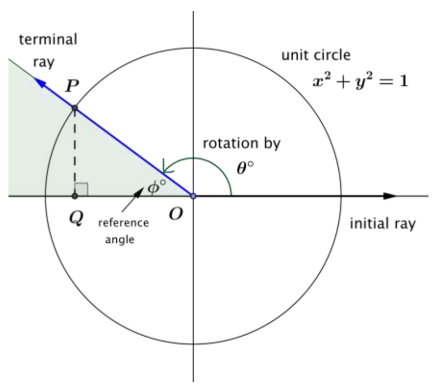
Notice that the $150°$ angle formed by $\overrightarrow{OP}$ and $\overrightarrow{OE}$ is exterior to the right triangle $\triangle POQ$. Angle POQ is the reference angle for rotation by $150°$. We can use symmetry and the fact that we know the sine and cosine ratios of $30°$ to find the values of the sine and cosine functions for 150 degrees of rotation.

- What are the coordinates (x_θ, y_θ) of point P?
  - *Using symmetry, we see that the y-coordinate of P is the same as it was for a $30°$ rotation but that the x-coordinate is the opposite sign as it was for a $30°$ rotation. Thus $(x_\theta, y_\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.*
- What is $\sin(150°)$?
  - $\sin(150°) = \frac{1}{2}$
- What is $\cos(150°)$?
  - $\cos(150°) = -\frac{\sqrt{3}}{2}$

Discussion (8 minutes)

- In general if we rotate the initial ray through more than $90°$, then the reference angle is the acute angle formed by the terminal ray and the x-axis. In the following diagrams, the measure of the reference angle is denoted by ϕ, the Greek letter phi. Let's start with the case where the terminal ray is rotated into the second quadrant.





> *Scaffolding:*
>
> To provide support with the term *reference angle*, have students create a graphic organizer in which they divide the page into four quadrants, draw a unit circle in the 1st quadrant with the terminal ray in Quadrant I, draw a unit circle in the 2nd quadrant with the terminal ray in Quadrant II, etc. Have students shade in the interior of the reference angles in all four cases before proceeding.

- If $90 < \theta < 180$, then the terminal ray of the rotation by $\theta°$ lies in the second quadrant. The reference angle formed by the terminal ray and the x-axis has measure $\phi°$ and is shaded in green in the figure on the right above. How does ϕ relate to θ?
  - $\phi = 180 - \theta$
- If we let Q be the foot of the perpendicular from P to the x-axis, then $\triangle OPQ$ is a right triangle. How can we find the lengths OQ and PQ?
  - *We can use triangle trigonometry: $OQ = \cos(\phi°)$, and $PQ = \sin(\phi°)$.*

> *Scaffolding:*
>
> For students not quite ready for this level of abstraction, use $\theta = 135$ for this discussion instead of a generic number θ.

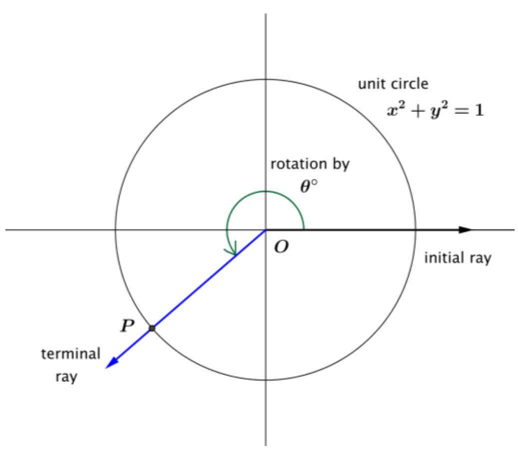
- How can we use these lengths to find the coordinates of point P?
  - *Since the x-coordinate of P is negative, and O is the origin, then the x-coordinate of P is $-OQ = -\cos(\phi°)$. Since the y-coordinate of P is positive, then the y-coordinate of P is $PQ = \sin(\phi°)$.*
- Ask students the following question to summarize these results: If $90 < \theta < 180$, then rotation by θ degrees places P in the second quadrant, with reference angle of measure ϕ degrees. Then what are the values of $\cos(\theta°)$ and $\sin(\theta°)$?
  - $\cos(\theta°) = -\cos(\phi°)$
  - $\sin(\theta°) = \sin(\phi°)$
- For example, what is $\cos(135°)$?
  - $\cos(135°) = -\cos(45°) = -\frac{\sqrt{2}}{2}$
- What is $\sin(135°)$?
  - $\sin(135°) = \sin(45°) = \frac{\sqrt{2}}{2}$

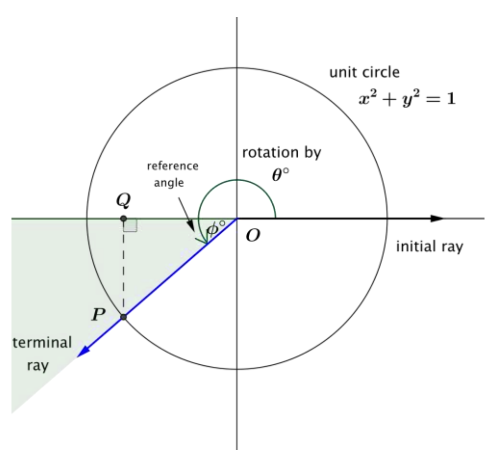
Ask students to turn to their neighbor or partner and summarize the main points of the previous discussion. Ask for a volunteer to present their summary to the class.

- *The sine and cosine of a degree measure that rotates point P outside of the first quadrant can be found by looking at the sine and cosine of the measure of the reference angle. We can find coordinates of point P by looking at the sine and cosine for the measure of the reference angle and then assign negative signs where the coordinate would be negative.*

Assign half of the class to work on the case when the terminal ray is located in the third quadrant and the other half to work on the case when the terminal ray is located in the fourth quadrant, or lead the whole class in a discussion for these cases. In either case, be sure to summarize the results for the remaining two quadrants.

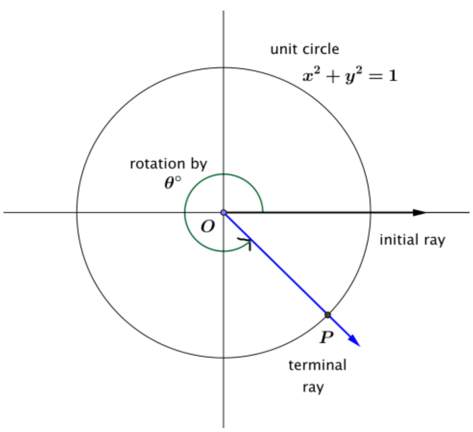
- In the diagram below, $180 < \theta < 270$, so that point P is in the third quadrant. Then, we know that both the x-coordinate and y-coordinate of P are negative.

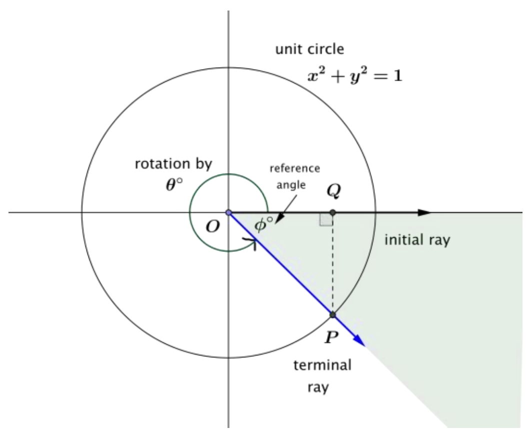




- If $180 < \theta < 270$, then the terminal ray of the rotation by $\theta°$ lies in the third quadrant. The reference angle formed by the terminal ray and the x-axis has measure $\phi°$ and is shaded in green in the figure on the right above. How does ϕ relate to θ?
  - $\phi = \theta - 180$

- If we let Q be the foot of the perpendicular from P to the x-axis, then $\triangle\ OPQ$ is a right triangle. How can we find the lengths OQ and PQ?
  - *We can use triangle trigonometry:* $OQ = \cos(\phi°)$ *and* $PQ = \sin(\phi°)$.
- How can we use these lengths to find the coordinates of point P?
  - *Since the x-coordinate of P is negative, and O is the origin, then the x-coordinate of P is $-OQ = -\cos(\phi°)$. Since the y-coordinate of P is also negative, then the y-coordinate of P is $-PQ = -\sin(\phi°)$.*
- Ask students the following question to summarize these results: If $180 < \theta < 270$, then rotation by θ degrees places P in the third quadrant, with reference angle of measure ϕ. Then what are the values of $\cos(\theta°)$ and $\sin(\theta°)$?
  - $\cos(\theta°) = -\cos(\phi°)$
  - $\sin(\theta°) = -\sin(\phi°)$
- For example, what is $\cos(225°)$?
  - $\cos(225°) = -\cos(45°) = -\frac{\sqrt{2}}{2}$
- What is $\sin(225°)$?
  - $\sin(225°) = -\sin(45°) = -\frac{\sqrt{2}}{2}$
- In the diagram below, $270 < \theta < 360$, so that point P is in the fourth quadrant. Then, we know that both the x-coordinate and y-coordinate of P are negative.





- If $270 < \theta < 360$, then the terminal ray of the rotation by $\theta°$ lies in the fourth quadrant. The reference angle formed by the terminal ray and the x-axis has measure $\phi°$ and is shaded in green in the figure on the right above. How does ϕ relate to θ?
  - $\phi = 360 - \theta$
- Again, if we let Q be the foot of the perpendicular from P to the x-axis, then $\triangle\ OPQ$ is a right triangle.
- How can we use the lengths OQ and PQ to find the coordinates of point P?
  - *Since the x-coordinate of P is positive, and O is the origin, then the x-coordinate of P is $OQ = \cos(\phi°)$. Since the y-coordinate of P is negative, then the y-coordinate of P is $-PQ = -\sin(\phi°)$.*

> *Scaffolding:*
>
> For students not quite ready for this level of abstraction, use $\theta = 225$ for this discussion instead of a generic number θ.

- Ask students the following question to summarize these results: If $270 < \theta < 360$, then rotation by θ degrees places P in the fourth quadrant, with reference angle of measure ϕ degrees. Then what are the values of $\cos(\theta°)$ and $\sin(\theta°)$?
  - $\cos(\theta°) = \cos(\phi°)$
  - $\sin(\theta°) = -\sin(\phi°)$
- For example, what is $\cos(315°)$?
  - $\cos(315°) = \cos(45°) = \frac{\sqrt{2}}{2}$
- What is $\sin(315°)$?
  - $\sin(315°) = -\sin(45°) = -\frac{\sqrt{2}}{2}$

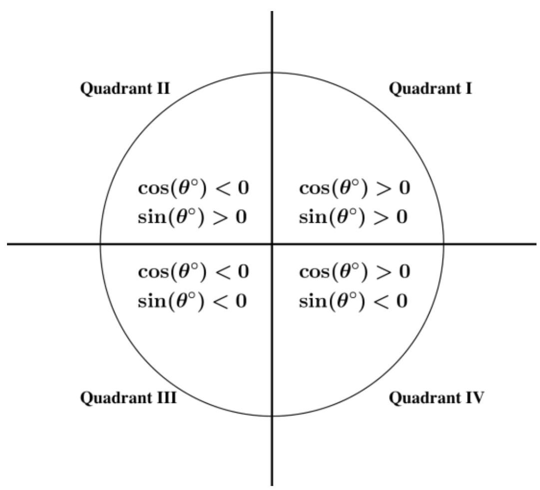
What we have just concluded is very important. We have just extended the definitions of sine and cosine from Geometry to almost any number of degrees of rotation between 0 and 360, when they were previously only defined for $0 < \theta < 90$. Lesson 5 extends the domain of the sine and cosine even further by exploring what happens if $\theta > 360$ and what happens if $\theta \leq 0$.

Discussion (2 minutes)

Ask students to discuss the following question with their neighbor. After a minute of discussion, lead students in completing the diagram below to indicate the positive and negative signs of the sine and cosine functions in the four quadrants of the coordinate plane.

- How do you know whether $\cos(\theta°)$ and $\sin(\theta°)$ are positive or negative in each quadrant?

Discussion



Exercises 3–5 (4 minutes)

These exercises serve to extend our working definition of sine and cosine from $0 < \theta < 90$ to most numbers of degrees of rotation θ such that $0 < \theta < 360$. Have students complete these exercises in pairs while the teacher circulates around the room and models as necessary.

Exercises 3–5

3. Suppose that P is the point on the unit circle obtained by rotating the initial ray counterclockwise through 120 degrees. Find the measure of the reference angle for $120°$, and then find $\sin(120°)$ and $\cos(120°)$.

   The measure of the reference angle for $120°$ is $60°$, and P is in Quadrant II. We have $\sin(120°) = \frac{\sqrt{3}}{2}$ and $\cos(120°) = -\frac{1}{2}$.

4. Suppose that P is the point on the unit circle obtained by rotating the initial ray counterclockwise through $240°$. Find the measure of the reference angle for $240°$, and then find $\sin(240°)$ and $\cos(240°)$.

   The measure of the reference angle for $240°$ is $60°$, and P is in Quadrant III. We have $\sin(240°) = -\frac{\sqrt{3}}{2}$ and $\cos(240°) = -\frac{1}{2}$.

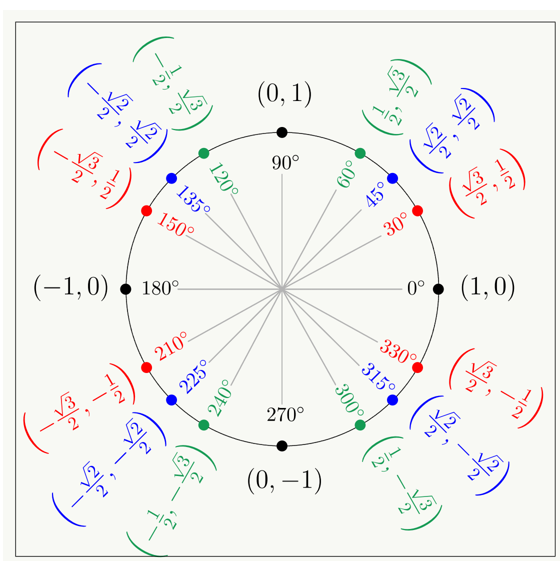
5. Suppose that P is the point on the unit circle obtained by rotating the initial ray counterclockwise through 330 degrees. Find the measure of the reference angle for $330°$, and then find $\sin(330°)$ and $\cos(330°)$.

   The measure of the reference angle for $330°$ is $30°$, and P is in Quadrant IV. We have $\sin(330°) = -\frac{1}{2}$ and $\cos(330°) = \frac{\sqrt{3}}{2}$.

Discussion (2 minutes)

- We have now made sense of the sine and cosine functions nearly all values of theta with $0 < \theta < 360$, where θ is measured in degrees. In the next lesson, we extend the domains of these two functions even further, so that they are be defined for any real number θ.
- The values of the sine and cosine functions at rotations of 30, 45, and 60 degrees and multiples of these rotations come up often in trigonometry. The diagram below summarizes the coordinates of these commonly referenced points.

Discussion



- Use the diagram to find $\cos(120°)$.
  - $-\frac{1}{2}$
- Use the diagram to find $\sin(300°)$.
  - $-\frac{\sqrt{3}}{2}$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements:

Lesson Summary

In this lesson we formalized the idea of the height and co-height of a Ferris wheel and defined the sine and cosine functions that give the x- and y-coordinates of the intersection of the unit circle and the initial ray rotated through θ degrees, for most values of θ with $0 < \theta < 360$.

- **The value of $\cos(\theta°)$ is the x-coordinate of the intersection point of the terminal ray and the unit circle.**
- **The value of $\sin(\theta°)$ is the y-coordinate of the intersection point of the terminal ray and the unit circle.**
- **The sine and cosine functions have domain of all real numbers and range $[-1, 1]$.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 4: From Circle-ometry to Trigonometry

Exit Ticket

1. How did we define the sine function for a number of degrees of rotation θ, where $0 < \theta < 360$?

2. Explain how to find the value of $\sin(210°)$ without using a calculator.

Exit Ticket Sample Solutions

1. **How did we define the sine function for a number of degrees of rotation θ, where $0 < \theta < 360$?**

   First we rotate the initial ray counterclockwise through θ degrees and find the intersection of the terminal ray with the unit circle. This intersection is point P. The y-coordinate of point P is the value of $\sin(\theta°)$.

2. **Explain how to find the value of $\sin(210°)$ without using a calculator.**

   The reference angle for and angle of measure $210°$ has measure $30°$, and a rotation by $210°$ counterclockwise places the terminal ray in the 3rd quadrant, where both coordinates of the intersection point P are negative.
   So, $\sin(210°) = -\sin(30°) = -\frac{1}{2}$.

Problem Set Sample Solutions

1. **Fill in the chart. Write in the reference angles and the values of the sine and cosine functions for the indicated values of θ.**

Amount of rotation, θ, in degrees	Measure of Reference Angle, in degrees	$\cos(\theta°)$	$\sin(\theta°)$
120	60	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
135	45	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
150	30	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
225	45	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
240	60	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
300	60	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
330	30	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

2. Using geometry, Jennifer correctly calculated that $\sin(15°) = \frac{1}{2}\sqrt{2-\sqrt{3}}$. Based on this information, fill in the chart.

Amount of rotation, θ, in degrees	Measure of Reference Angle, in degrees	$\cos(\theta°)$	$\sin(\theta°)$
15	15	$\frac{1}{2}\sqrt{2+\sqrt{3}}$	$\frac{1}{2}\sqrt{2-\sqrt{3}}$
165	15	$-\frac{1}{2}\sqrt{2+\sqrt{3}}$	$\frac{1}{2}\sqrt{2-\sqrt{3}}$
195	15	$-\frac{1}{2}\sqrt{2+\sqrt{3}}$	$-\frac{1}{2}\sqrt{2-\sqrt{3}}$
345	15	$\frac{1}{2}\sqrt{2+\sqrt{3}}$	$-\frac{1}{2}\sqrt{2-\sqrt{3}}$

3. Suppose $0 < \theta < 90$ and $\sin(\theta°) = \frac{1}{\sqrt{3}}$. What is the value of $\cos(\theta°)$?

$\cos(\theta°) = \frac{\sqrt{6}}{3}$

4. Suppose $90 < \theta < 180$ and $\sin(\theta°) = \frac{1}{\sqrt{3}}$. What is the value of $\cos(\theta°)$?

$\cos(\theta°) = -\frac{\sqrt{6}}{3}$

5. If $\cos(\theta°) = -\frac{1}{\sqrt{5}}$, what are two possible values of $\sin(\theta°)$?

$\sin(\theta°) = \frac{2}{\sqrt{5}}$ *or* $\sin(\theta°) = -\frac{2}{\sqrt{5}}$

6. Johnny rotated the initial ray through θ degrees, found the intersection of the terminal ray with the unit circle, and calculated that $\sin(\theta°) = \sqrt{2}$. Ernesto insists that Johnny made a mistake in his calculation. Explain why Ernesto is correct.

Johnny must have made a mistake since the sine of a number cannot be greater than 1.

7. If $\sin(\theta°) = 0.5$, and we know that $\cos(\theta°) < 0$, then what is the smallest possible positive value of θ?

150

8. The vertices of $\triangle ABC$ have coordinates $A(0,0)$, $B(12,5)$, and $C(12,0)$.

   a. Argue that $\triangle ABC$ is a right triangle.

   Clearly $\overline{AC}$ *is horizontal and* $\overline{BC}$ *is vertical, so* $\triangle ABC$ *is a right triangle.*

   b. What are the coordinates where the hypotenuse of $\triangle ABC$ intersects the unit circle $x^2 + y^2 = 1$?

   Using similar triangles, the hypotenuse intersects the unit circle at $\left(\frac{12}{13}, \frac{5}{13}\right)$.

   c. Let θ denote the number of degrees of rotation from $\overrightarrow{AC}$ to $\overrightarrow{AB}$. Calculate $\sin(\theta°)$ and $\cos(\theta°)$.

   By the answer to part (b), $\sin(\theta°) = \frac{5}{13}$, *and* $\cos(\theta°) = \frac{12}{13}$.

9. The vertices of $\triangle ABC$ have coordinates $A(0,0)$, $B(4,3)$, and $C(4,0)$. The vertices of $\triangle ADE$ are $A(0,0)$, $D(3,4)$, and $E(3,0)$.

a. Argue that $\triangle ABC$ is a right triangle.

Clearly $\overline{AC}$ is horizontal and $\overline{BC}$ is vertical, so $\triangle ABC$ is a right triangle.

b. What are the coordinates where the hypotenuse of $\triangle ABC$ intersects the unit circle $x^2 + y^2 = 1$?

Using similar triangles, the hypotenuse intersects the unit circle at $\left(\frac{4}{5}, \frac{3}{5}\right)$.

c. Let θ denote the number of degrees of rotation from $\overrightarrow{AC}$ to $\overrightarrow{AB}$. Calculate $\sin(\theta°)$ and $\cos(\theta°)$.

By the answer to part (b), $\sin(\theta°) = \frac{3}{5}$, and $\cos(\theta°) = \frac{4}{5}$.

d. Argue that $\triangle ADE$ is a right triangle.

Since $a^2 + b^2 = c^2$, the converse of the Pythagorean theorem guarantees that $\triangle ADE$ is a right triangle.

e. What are the coordinates where the hypotenuse of $\triangle ADE$ intersects the unit circle $x^2 + y^2 = 1$?

Using similar triangles, the hypotenuse intersects the unit circle at $\left(\frac{3}{5}, \frac{4}{5}\right)$.

f. Let ϕ denote the number of degrees of rotation from $\overrightarrow{AE}$ to $\overrightarrow{AD}$. Calculate $\sin(\phi°)$ and $\cos(\phi°)$.

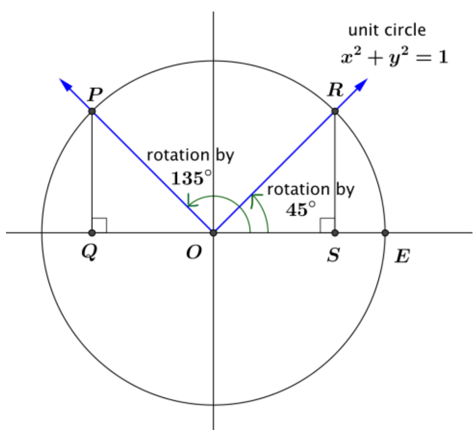
By the answer to part (e), $\sin(\phi°) = \frac{4}{5}$, and $\cos(\phi°) = \frac{3}{5}$.

g. What is the relation between the sine and cosine of θ and the sine and cosine of ϕ?

We find that $\sin(\phi°) = \frac{4}{5} = \cos(\theta°)$, and $\cos(\phi°) = \frac{3}{5} = \sin(\theta°)$.

10. Use a diagram to explain why $\sin(135°) = \sin(45°)$, but $\cos(135°) \neq \cos(45°)$.

Let O be the center of the circle, let P and R be the points where the terminal rays of rotation by $135°$ and $45°$, and let Q and S be the feet of the perpendicular lines from P and R to the x-axis, respectively. Then $\triangle OPQ$ and $\triangle ORS$ are both isosceles right triangles with hypotenuses of length 1, so they are congruent. Thus, $PQ = RS$, and $OQ = OS$. Let the coordinates of P and R be (x_P, y_P) and (x_R, y_R). Then $x_P = -OQ = -OR = -x_R$, and $y_P = PQ = RS = y_Q$. Then we have $\cos(135°) = -\cos(45°)$, and $\sin(135°) = \sin(45°)$.



Lesson 5: Extending the Domain of Sine and Cosine to All Real Numbers

Student Outcomes

- Students define sine and cosine as functions for all real numbers measured in degrees.
- Students evaluate the sine and cosine functions at multiples of 30 and 45.

Lesson Notes

In the preceding lesson, students extended the previous definition of sine and cosine from $0 < \theta < 90$ to $0 < \theta < 360$ using right triangle trigonometry, connecting the sine function to the height function of the Ferris wheel and the cosine function to the co-height function. In this lesson, students extend the domain of the sine and cosine functions to the entire real number line, at which point a complete definition of these two functions can finally be provided. Students continue to use the context of the Ferris wheel to understand the implications of counterclockwise rotation through $\theta \geq 360$ and clockwise rotation through $\theta \leq 0$.

As with the previous lesson, a theoretical understanding of the process of extending the sine and cosine functions to the entire real line is being developed, so calculators should not be allowed for any part of this lesson, including the problem set. Rotations in this lesson are restricted to multiples of $30°$ or $45°$, so the focus of this lesson is assigning the proper positive or negative signs to the value of the sine and cosine functions.

Opening Exercise (4 minutes)

The Opening Exercises serve to remind students of the concept of a remainder and lead into finding sine and cosine for a number of degrees of rotation greater than 360. While the context of these exercises is artificial, it leads students to think about the amount leftover after a rotation of more than one full turn, which they need in the upcoming tasks.

Allow students to work individually or in pairs on the following division problems:

Opening Exercise

a. **Suppose that a group of 360 coworkers pool their money, buying a single lottery ticket every day with the understanding that if any ticket is a winning ticket, the group will split the winnings evenly, and they will donate any leftover money to the local high school. Using this strategy, if the group wins $\$1,000$, how much money will be donated to the school?**

Since $\$1,000 = \$2(360) + \$280$, each coworker wins $\$2$, and the local school receives the leftover $\$280$.

b. **What if the winning ticket is worth $\$250,000$? Using the same plan as in part (a), how much money will be donated to the school?**

Since $\$250,000 = \$694(360) + \$160$, each coworker wins $\$694$, and the school receives the leftover $\$160$.

c. **What if the winning ticket is worth $\$540,000$? Using the same plan as in part (a), how much money will be donated to the school?**

Since $\$540,000 = \$1,500(360) + \$0$, each coworker wins $\$1,500$, and the school receives nothing.

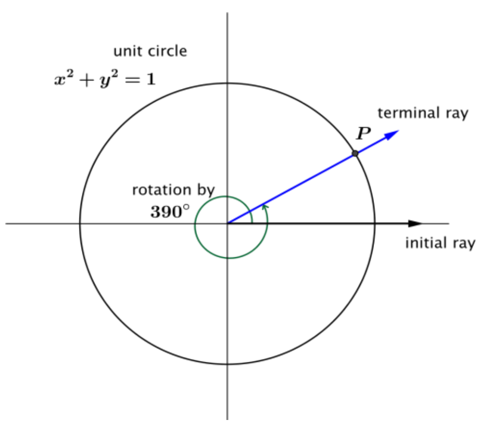
Discussion (3 minutes)

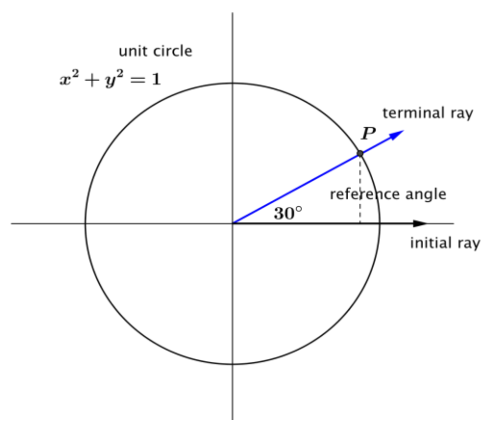
- During yesterday's lesson, we found a way to calculate the sine and cosine functions for rotations of the initial ray (made from the positive x-axis) through θ degrees, for $0 < \theta < 360$. Today, we investigate what happens if θ takes on a value outside of the interval $(0, 360)$. Remember that our motivating examples for the sine and cosine functions were the height and co-height functions associated with a rotating Ferris wheel. Let's return to that context for this discussion.
- In reality, a Ferris wheel doesn't just go around once and then stop. It rotates a number of times and then stops to let the riders off. How can we extend our ideas about sine and cosine to a counterclockwise rotation through more than $360°$?
  - *Ask for ideas from the class. Guide them to notice the periodicity of rotation about the origin; once a rotation passes $360°$, the position of the point P starts over.*

Example 1 (4 minutes)

Suppose that P is the point on the unit circle obtained from rotating the initial ray through $390°$. Find $\sin(390°)$ and $\cos(390°)$.

- Does it make sense to think of a reference angle for this rotation?
  - *Yes because $390 = 360 + 30$.*





- What is the measure of the reference angle for this rotation?
  - *The reference angle is a $30°$ angle.*
- What are the coordinates (x_θ, y_θ) of point P?
  - *The terminal ray lands in the same place after a $390°$ rotation as it did after a $30°$ rotation. Thus, point P is at the same location as if we had only rotated by $30°$. Thus, we have*

    $$(x_\theta, y_\theta) = (\cos(30°), \sin(30°)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

- What is $\sin(390°)$?
  - $\sin(390°) = \frac{1}{2}$
- What is $\cos(390°)$?
  - $\cos(390°) = \frac{\sqrt{3}}{2}$

Exercises 1–5 (7 minutes)

Allow students to work in pairs or small groups on these exercises. Do not allow the use of calculators. Circulate around the room while students are working and remind them to think about remainders, as they did in the Opening Exercise.

Exercises 1–5

1. **Find $\cos(405°)$ and $\sin(405°)$. Identify the measure of the reference angle.**

   Since $405 = 360 + 45$, and a $45°$ rotation places the terminal ray in the first quadrant, the reference angle measures $45°$. Then, we have $\cos(405°) = \cos(45°) = \frac{\sqrt{2}}{2}$, and $\sin(405°) = \sin(45°) = \frac{\sqrt{2}}{2}$.

2. **Find $\cos(840°)$ and $\sin(840°)$. Identify the measure of the reference angle.**

   Since $840 = 2(360) + 120$, and a $120°$ rotation places the terminal ray in the second quadrant, the reference angle measures $60°$. Then, we have $\cos(840°) = -\cos(60°) = -\frac{1}{2}$, and $\sin(840°) = \sin(60°) = \frac{\sqrt{3}}{2}$.

3. **Find $\cos(1680°)$ and $\sin(1680°)$. Identify the measure of the reference angle.**

   Since $1680 = 4(360) + 240$, and a $240°$ rotation places the terminal ray in the third quadrant, the reference angle measures $60°$. Then, we have $\cos(840°) = -\cos(60°) = -\frac{1}{2}$, and $\sin(840°) = -\sin(60°) = -\frac{\sqrt{3}}{2}$.

4. **Find $\cos(2115°)$ and $\sin(2115°)$. Identify the measure of the reference angle.**

   Since $2115 = 5(360) + 315$, and a $315°$ rotation places the terminal ray in the fourth quadrant, the reference angle measures $45°$. Then, we have $\cos(2115°) = \cos(45°) = \frac{\sqrt{2}}{2}$, and $\sin(2115°) = -\sin(45°) = -\frac{\sqrt{2}}{2}$.

5. **Find $\cos(720\,030°)$ and $\sin(720\,030°)$. Identify the measure of the reference angle.**

   Since $720\,030 = 2000(360) + 30$, and a $30°$ rotation places the terminal ray in the first quadrant, the reference angle measures $30°$. Then, we have $\cos(720\,030°) = \cos(30°) = \frac{\sqrt{3}}{2}$, and $\sin(720\,030°) = \sin(30°) = \frac{1}{2}$.

Scaffolding:

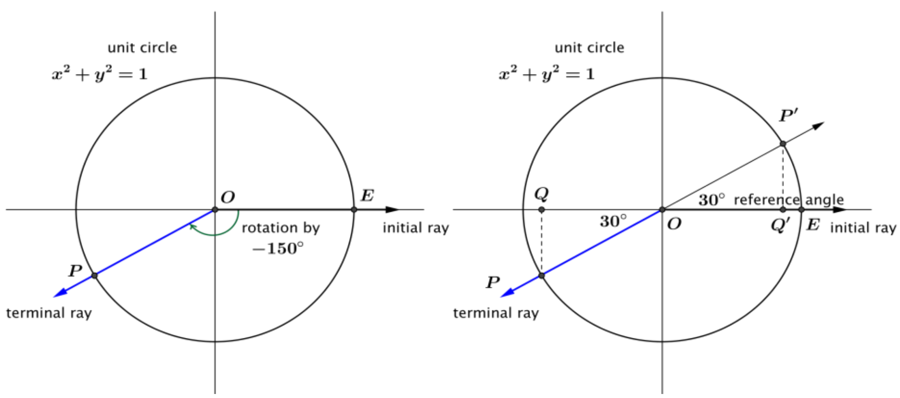
- Remind students to draw pictures of the terminal ray and the reference angle.
- Ask struggling students to think about how many times the ray is rotated around a full circle before coming to a stop.

Discussion (2 minutes)

- Now we know how to calculate the values of the sine and cosine functions for rotating further than $360°$ counterclockwise. But what if the Ferris wheel malfunctions and starts rotating backward? Does it still make sense to talk about the height and co-height functions if the Ferris wheel is turning the wrong way?
  - *Solicit ideas from the class. Guide them to realize that the height and co-height functions only depend on the final position of the point after the rotation, not the direction in which the wheel was rotated. Thus, it makes perfect sense to define sine and cosine functions for a ray rotating backward around the circle.*
- In our definition of sine and cosine, how can we indicate that the rotation is happening in the opposite direction from our normal counterclockwise rotation?
  - *We use a negative sign to indicate rotation in the clockwise direction. That is, $\theta = -60$ indicates a clockwise rotation by $60°$.*

Example 2 (3 minutes)

Suppose that P is the point on the unit circle obtained from rotating the initial ray through $-150°$. Find $\sin(-150°)$ and $\cos(-150°)$.



a. What is the measure of the reference angle for $\angle POE$?

The reference angle is $\angle POQ$, which has measure $30°$ since $180 - 150 = 30$.

b. What are the coordinates (x_θ, y_θ) of point P?

Point P lands in the same place after the initial ray is rotated by $150°$ clockwise as it did after a $210°$ counterclockwise rotation. Thus, $(x_\theta, y_\theta) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

c. What is $\sin(-150°)$?

$\sin(-150°) = -\frac{1}{2}$

d. What is $\cos(-150°)$?

$$\cos(-150°) = -\frac{\sqrt{3}}{2}$$

Exercises 6–10 (6 minutes)

Allow students to work in pairs or small groups on these exercises. Allow only the use of calculators without trigonometric capabilities; for example, it might be helpful to use a calculator to express -2205 as $-6(360) - 45$. Circulate around the room while students are working and remind them to think about writing a rotation in terms of whole $360°$ rotations, beginning with Exercise 8.

Exercises 6–10

6. **Find $\cos(-30°)$ and $\sin(-30°)$. Identify the measure of the reference angle.**

   Since a – $30°$ rotation places the terminal ray in the fourth quadrant, the reference angle measures $30°$. Then, we have $\cos(-30°) = \cos(30°) = \frac{\sqrt{3}}{2}$, and $\sin(-30°) = -\sin(30°) = -\frac{1}{2}$.

7. **Find $\cos(-135°)$ and $\sin(-135°)$. Identify the measure of the reference angle.**

   Since the terminal ray of a $-135°$ rotation aligns with the terminal ray of a $225°$ rotation in the third quadrant, the reference angle measures $45°$. Then, we have $\cos(-135°) = -\cos(45°) = -\frac{\sqrt{2}}{2}$, and $\sin(-135°) = -\sin(45°) = -\frac{\sqrt{2}}{2}$.

8. **Find $\cos(-1320°)$ and $\sin(-1320°)$. Identify the measure of the reference angle.**

   Since the terminal ray of a $-1320°$ rotation aligns with the terminal ray of a $120°$ rotation in the second quadrant, the reference angle measures $60°$. Then, we have $\cos(-1320°) = -\cos(60°) = -\frac{1}{2}$, and $\sin(-1320°) = \sin(60°) = \frac{\sqrt{3}}{2}$.

9. **Find $\cos(-2205°)$ and $\sin(-2205°)$. Identify the measure of the reference angle.**

   Since the terminal ray of a $-2205°$ rotation aligns with the terminal ray of a $-45°$ rotation in the fourth quadrant, the reference angle measures $45°$. Then, we have $\cos(-2205°) = \cos(45°) = \frac{\sqrt{2}}{2}$, and $\sin(-2205°) = -\sin(45°) = -\frac{\sqrt{2}}{2}$.

10. **Find $\cos(-2835°)$ and $\sin(-2835°)$. Identify the measure of the reference angle.**

    Since the terminal ray of a $-2835°$ aligns with the terminal ray of a $45°$ rotation in the first quadrant, the reference angle measures $45°$. Then, we have $\cos(-2835°) = -\cos(45°) = \frac{\sqrt{2}}{2}$, and $\sin(-2835°) = \sin(45°) = \frac{\sqrt{2}}{2}$.

Scaffolding:

- Remind students to draw pictures of the terminal ray and the reference angle.
- To help find the reference angle, ask students to count the number of whole rotations and then find the remaining degrees of rotation. Then, have them apply the techniques of Lesson 4 to find the reference angle.

Discussion (2 minutes)

- At this point, we have defined the sine and cosine functions for almost any positive or negative rotation, but there are a few cases we have not yet addressed. What if the Ferris wheel completely breaks down and will not move at all once you have been loaded into your car? Does it still make sense to talk about the height and co-height functions if the Ferris wheel never gets started? Can we still think of the car as rotating through a number of degrees?
  - *If the Ferris wheel never moves, then point P has technically rotated through $0°$. In this case, our position starts and ends at point P with coordinates $(x_\theta, y_\theta) = (1,0)$. We then have $\sin(0°) = 0$ and $\cos(0°) = 1$, which makes sense since the height hasn't changed because the machine is not working.*

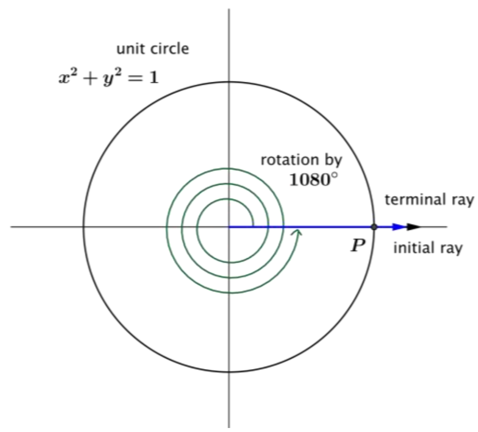
Discussion (7 minutes)

If students benefit from repetition, choose to model all four of the cases in this discussion. Otherwise, model the first case, and assign student groups to work through the remaining three cases and report back to the class. When modeling these cases, allow students a few minutes to sketch the rotation and try to find the reference angle, and then begin the discussion.

Discussion

Case 1: What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the positive x-axis, such as $1080°$?

Our definition of a reference angle is the angle formed by the terminal ray and the x-axis, but our terminal ray lies along the x-axis, so the terminal ray and the x-axis form a zero angle.



How would we assign values to $\cos(1080°)$ and $\sin(1080°)$?

Use the coordinates of point P, which are $(1, 0)$. Then, $\cos(1080°) = 1$, and $\sin(1080°) = 0$.

What if we rotated around $24000°$, which is 400 turns? What are $\cos(24000°)$ and $\sin(24000°)$?

The terminal ray is the same as it was for $0°$, so the intersection point P has coordinates $(1, 0)$. Thus, $\cos(24000°) = 1$, and $\sin(24000°) = 0$.

State a generalization of these results:

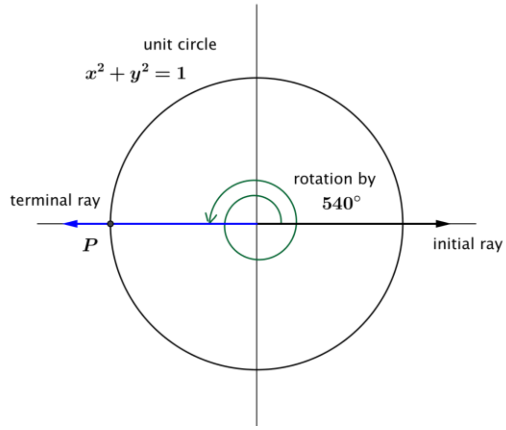
If $\theta = n \cdot 360$, for some integer n, then $\cos(\theta°) =$ ____, and $\sin(\theta°) =$ _____.

If $\theta = n \cdot 360$, for some integer n, then $\cos(\theta°) = 1$, and $\sin(\theta°) = 0$.

Case 2: What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the negative x-axis, such as $540°$?

How would we assign values to $\cos(540°)$ and $\sin(540°)$?

Use the coordinates of point P, which are $(-1, 0)$.
Then $\cos(540°) = -1$, and $\sin(540°) = 0$.



What are the values of $\cos(900°)$ and $\sin(900°)$? How do you know?

Since the terminal ray of rotation by $900°$ lies along the negative x-axis, it coincides with the terminal ray of rotation by $540°$. Thus, the coordinates of the intersection point P are $(-1, 0)$, and we have $\cos(900°) = -1$, and $\sin(900°) = 0$.

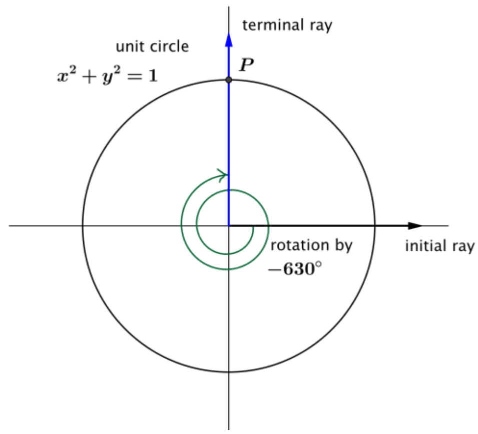
State a generalization of these results:

If $\theta = n \cdot 360 + 180$, for some integer n, then $\cos(\theta°) =$ ____, and $\sin(\theta°) =$ _____.

If $\theta = n \cdot 360 + 180$, for some integer n, then $\cos(\theta°) = -1$, and $\sin(\theta°) = 0$.

Case 3: What about the values of the sine and cosine function for rotations that are $90°$ more than a number of full turns, such as $-630°$? How would we assign values to $\cos(-630°)$, and $\sin(-630°)$?

Use the coordinates of point P, which are $(0, 1)$.
Then, $\cos(-630°) = 0$, and $\sin(-630°) = 1$.



Can we generalize to any rotation that produces a terminal ray along the positive y-axis?

Yes

State a generalization of these results:

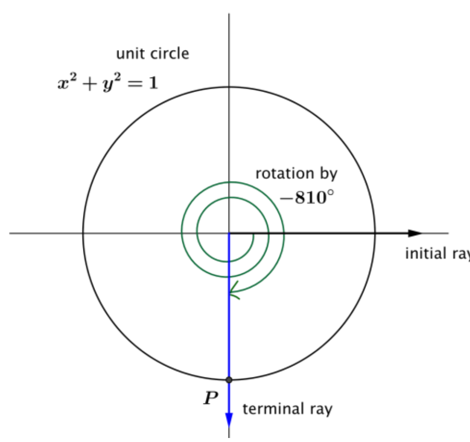
If $\theta = n \cdot 360 + 90$, for some integer n, then $\cos(\theta°) =$ ____, and $\sin(\theta°) =$ _____.

If $\theta = n \cdot 360 + 90$, for some integer n, then $\cos(\theta°) = 0$, and $\sin(\theta°) = 1$.

Case 4: What about the values of the sine and cosine function for rotations whose terminal ray lies along the negative y-axis, such as $-810°$?

How would we assign values to $\cos(-810°)$ and $\sin(-810°)$?

Use the coordinates of point P, which are $(0, -1)$.
Then, $\cos(-810°) = 0$, and $\sin(-810°) = -1$.



Can we generalize to any rotation that produces a terminal ray along the negative y-axis?

Yes

State a generalization of these results:

If $\theta = n \cdot 360 + 270$, for some integer n, then, $\cos(\theta°) =$ ____, and $\sin(\theta°) =$ _____.

If $\theta = n \cdot 360 + 270$, for some integer n, then $\cos(\theta°) = 0$, and $\sin(\theta°) = -1$.

Discussion (2 minutes)

Students have now made sense of the sine and cosine functions for any number of degrees of rotation, whether positive, negative, or zero. They are now ready to define sine and cosine as functions of any real number.

Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ) in the coordinate plane. The value of $\sin(\theta°)$ is y_θ, and the value of $\cos(\theta°)$ is x_θ.

- What is the domain of the sine function?
  - *The domain of the sine function is all real numbers.*
- What is the range of the sine function?
  - *The range of the sine function is* $[-1,1]$.
- What is the domain of the cosine function?
  - *The domain of the cosine function is all real numbers.*
- What is the range of the cosine function?
  - *The range of the cosine function is* $[-1,1]$.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements:

Lesson Summary

In this lesson the definition of the sine and cosine are formalized as functions of a number of degrees of rotation, θ. The initial ray made from the positive x-axis through θ degrees is rotated, going counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$. The point P is defined by the intersection of the terminal ray and the unit circle.

1. **The value of $\cos(\theta°)$ is the x-coordinate of P.**
2. **The value of $\sin(\theta°)$ is the y-coordinate of P.**
3. **The sine and cosine functions have domain of all real numbers and range $[-1, 1]$.**

Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 5: Extending the Domain of Sine and Cosine to All Real Numbers

Exit Ticket

1. Calculate $\cos(480°)$ and $\sin(480°)$.

2. Explain how we calculate the sine and cosine functions for a value of θ so that $540 < \theta < 630$.

Exit Ticket Sample Solutions

1. Calculate $\cos(480°)$ and $\sin(480°)$.

   Since $480° = 360° + 120°$, the terminal ray of the rotated initial ray is in the 2nd quadrant. The reference angle is a $60°$ angle, so we have $\cos(480°) = -\cos(60°) = -\frac{1}{2}$, and $\sin(480°) = \sin(60°) = \frac{\sqrt{3}}{2}$.

2. Explain how we calculate the sine and cosine functions for a value of θ so that $540 < \theta < 630$.

   Since $540 < \theta < 630$, the terminal ray is in the 3rd quadrant. The reference angle is the angle formed by the terminal ray and the negative x-axis; let the reference angle have measure ϕ degrees. Thus, the sine and cosine of θ will be the opposite of the sine and cosine of ϕ: $\cos(\theta°) = -\cos(\phi°)$, and $\sin(\theta°) = \sin(\phi°)$.

Problem Set Sample Solutions

1. Fill in the chart. Write in the measures of the reference angles and the values of the sine and cosine functions for the indicated values of θ.

Number of degrees of rotation, θ	Quadrant	Measure of Reference Angle, in degrees	$\cos(\theta°)$	$\sin(\theta°)$
690	*IV*	30	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
810	*None*	90	0	1
1560	*II*	60	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
1440	*None*	0	1	0
855	*II*	45	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
-330	*I*	30	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
-4500	*None*	0	-1	0
-510	*III*	30	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
-135	*III*	45	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
-1170	*None*	90	0	-1

2. Using geometry, Jennifer correctly calculated that $\sin(15°) = \frac{1}{2}\sqrt{2-\sqrt{3}}$. Based on this information, fill in the chart:

Number of degrees of rotation, θ	Quadrant	Measure of Reference Angle, in degrees	$\cos(\theta°)$	$\sin(\theta°)$
525	*II*	15	$-\frac{1}{2}\sqrt{2+\sqrt{3}}$	$\frac{1}{2}\sqrt{2-\sqrt{3}}$
705	*IV*	15	$\frac{1}{2}\sqrt{2+\sqrt{3}}$	$-\frac{1}{2}\sqrt{2-\sqrt{3}}$
915	*III*	15	$-\frac{1}{2}\sqrt{2+\sqrt{3}}$	$-\frac{1}{2}\sqrt{2-\sqrt{3}}$
-15	*IV*	15	$\frac{1}{2}\sqrt{2+\sqrt{3}}$	$-\frac{1}{2}\sqrt{2-\sqrt{3}}$
-165	*III*	15	$-\frac{1}{2}\sqrt{2+\sqrt{3}}$	$-\frac{1}{2}\sqrt{2-\sqrt{3}}$
-705	*I*	15	$\frac{1}{2}\sqrt{2+\sqrt{3}}$	$\frac{1}{2}\sqrt{2-\sqrt{3}}$

3. Suppose θ represents a number of degrees of rotation and that $\sin(\theta°) = 0.5$. List the first six possible positive values that θ can take.

   $30, 150, 390, 510, 750, 870$

4. Suppose θ represents a number of degrees of rotation and that $\sin(\theta°) = -0.5$. List six possible negative values that θ can take.

   $-30, -150, -390, -510, -750, -870$

5. Suppose θ represents a number of degrees of rotation. Is it possible that $\cos(\theta°) = \frac{1}{2}$ and $\sin(\theta°) = \frac{1}{2}$?

   No. If $\cos(\theta°) = \frac{1}{2}$ and $\sin(\theta°) = \frac{1}{2}$, then the coordinates of point P are $\left(\frac{1}{2}, \frac{1}{2}\right)$, but this point doesn't lie on the unit circle.

6. Jane says that since the reference angle for a rotation through $-765°$ has measure $45°$, then $\cos(-765°) = \cos(45°)$, and $\sin(-765°) = \sin(45°)$. Explain why she is or is not correct.

   Jane is wrong. Because the terminal ray of the rotated initial ray lies in the fourth quadrant, we know that the y-coordinate changes sign. Thus, $\cos(-765°) = \cos(45°)$, but $\sin(-765°) = -\sin(45°)$.

7. Doug says that since the reference angle for a rotation through $765°$ has measure $45°$, then $\cos(765°) = \cos(45°)$, and $\sin(765°) = \sin(45°)$. Explain why he is or is not correct.

   Doug's conclusion is true, but his logic may be faulty. The reason $\cos(765°) = \cos(45°)$ and $\sin(765°) = \sin(45°)$ is that the terminal angle of the rotated ray lies in the first quadrant.

Lesson 6: Why Call It Tangent?

Student Outcomes

- Students define the tangent function and understand the historic reason for its name.
- Students use special triangles to determine geometrically the values of the tangent function for $30°$, $45°$, and $60°$.

Lesson Notes

MP.7 & MP.8

In this lesson, the right triangle definition of the tangent ratio of an acute angle θ, $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, is extended to the tangent function defined for all real numbers θ where $\cos(\theta°) \neq 0$. The word *tangent* already has geometric meaning, so the historical reasons for naming this particular function *tangent* are investigated. Additionally, the correlation of $\tan(\theta°)$ with the slope of the line that coincides with the terminal ray after rotation by θ degrees is noted. These three different interpretations of the tangent function can be used immediately to analyze properties and compute values of the tangent function. Students look for and make use of structure to develop the definitions in Exercises 7 and 8 and look for and express regularity in repeated reasoning using what they know about the sine and cosine functions applied to the tangent function in Exercise 3.

This lesson depends on vocabulary from Geometry such as secant lines and tangent lines. The terms provided are used for reference in this lesson and in subsequent lessons.

TANGENT FUNCTION (description): The *tangent function*,

$$\tan: \{x \in \mathbb{R} \mid x \neq 90 + 180k \text{ for all integers } k\} \to \mathbb{R}$$

can be defined as follows: Let θ be any real number such that $\theta \neq 90 + 180k$, for all integers k. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\tan(\theta°)$ is $\frac{y_\theta}{x_\theta}$.

The following trigonometric identity,

$$\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)} \text{ for all } \theta \neq 90 + 180k, \text{ for all integers } k,$$

or simply, $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, should be talked about almost immediately and used as the working definition of tangent.

SECANT TO A CIRCLE: A *secant line to a circle* is a line that intersects a circle in exactly two points.

TANGENT TO A CIRCLE: A *tangent line to a circle* is a line in the same plane that intersects the circle at one and only one point.

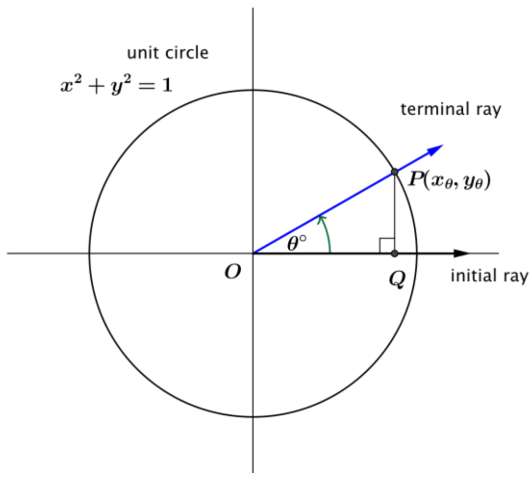
Classwork

Opening Exercise (4 minutes)

The Opening Exercise leads students to the description of $\tan(\theta°)$ as the quotient of y_θ and x_θ using right triangle trigonometry.

Opening Exercise

Let $P(x_\theta, y_\theta)$ be the point where the terminal ray intersects the unit circle after rotation by θ degrees, as shown in the diagram below.



Scaffolding:

For students not ready for this level of abstraction, use $\theta = 30$ for this example instead of the generic value θ.

a. **Using triangle trigonometry, what are the values of x_θ and y_θ in terms of θ?**

$x_\theta = \cos(\theta°)$, *and* $y_\theta = \sin(\theta°)$

b. **Using triangle trigonometry, what is the value of $\tan(\theta°)$ in terms of x_θ and y_θ?**

$\tan(\theta°) = \frac{y_\theta}{x_\theta}$

c. **What is the value of $\tan(\theta°)$ in terms of θ?**

$\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$

Discussion (6 minutes)

MP.3

In the previous lessons, the idea of the sine and cosine ratios of a triangle were extended to the sine and cosine functions of a real number, θ, that represents the number of degrees of rotation of the initial ray in the coordinate plane. In the following discussion, similarly the idea of the tangent ratio of an acute angle of a triangle is extended to the tangent function $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ on a subset of the real numbers.

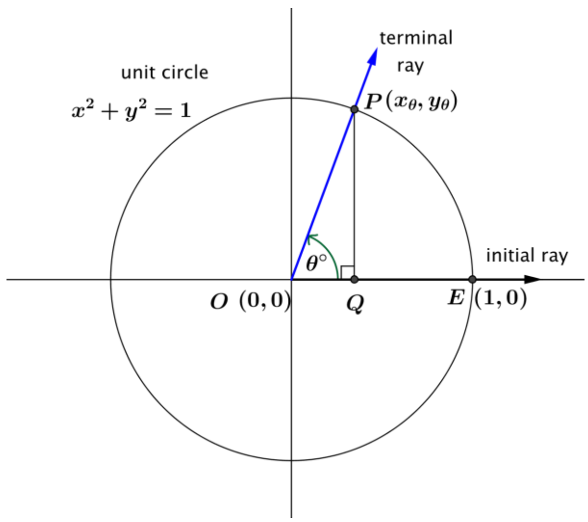
In this discussion, students should notice that the tangent ratio of an angle in a triangle does not extend to the entire real line because we need to avoid division by zero. Encourage students to find a symbolic representation for the points excluded from the domain of the tangent function; that is, the tangent function is defined for all real numbers θ except $\theta = 90 + 180k$, for all integers k.

As the discussion progresses, refer frequently to the image of the unit circle with the initial ray along the positive x-axis and the terminal ray intersecting the unit circle at a point P with coordinates (x_θ, y_θ), as was done in the Opening Exercise. Encourage students to draw similar diagrams in their own notes as well.

Discussion

A description of the tangent function is provided below. Be prepared to answer questions based on your understanding of this function and to discuss your responses with others in your class.

Let θ be any real number. In the Cartesian plane, rotate the nonnegative x-axis by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). If $x_\theta \neq 0$, then the value of $\tan(\theta°)$ is $\frac{y_\theta}{x_\theta}$. In terms of the sine and cosine functions, $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ for $\cos(\theta°) \neq 0$.



Scaffolding:

To recall some of the information students have developed in the last few lessons, drawing the unit circle on the board with a reference angle and sine and cosine labeled may be helpful. A picture is included.

- We have defined the tangent function to be the quotient $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ for $\cos(\theta°) \neq 0$. Why do we specify that $\cos(\theta°) \neq 0$?
  - *We cannot divide by zero, so the tangent function cannot be defined where the denominator is zero.*
- Looking at the unit circle in the figure, which segment has a measure equal to $\sin(\theta°)$, and which segment has a measure equal to $\cos(\theta°)$?
  - $PQ = \sin(\theta°)$, *and* $OQ = \cos(\theta°)$.
- Looking at the unit circle, identify several values of θ that cause $\tan(\theta°)$ to be undefined. (Scaffolding: When is the x-coordinate of point P zero?)
  - *When* $\cos(\theta°) = 0$, *then* $\tan(\theta°)$ *is undefined, which happens when the terminal ray is vertical so that point P lies along the y-axis. The following numbers of degrees of rotation locate the terminal ray along the y-axis:* $90, 270, -90, 450$.
- Describe all numbers θ for which $\cos(\theta°) = 0$.
  - $90 + 180k$, *for any integer* k

- How can we describe the domain of the tangent function, other than all real numbers θ with $\cos(\theta°) \neq 0$?
    - *The domain of the tangent function is all real numbers θ such that $\theta \neq 90 + 180k$, for all integers k.*

Exercise 1 (8 minutes)

Have students work in pairs or small groups to complete this table and answer the questions that follow. Then debrief the groups in a discussion.

Exercise 1

1. **For each value of θ in the table below, use the given values of $\sin(\theta°)$ and $\cos(\theta°)$ to approximate $\tan(\theta°)$ to two decimal places.**

θ (degrees)	$\sin(\theta°)$	$\cos(\theta°)$	$\tan(\theta°)$
-89.9	-0.999998	0.00175	-572.96
-89	-0.9998	0.0175	-57.29
-85	-0.996	0.087	-11.43
-80	-0.98	0.17	-5.67
-60	-0.87	0.50	-1.73
-40	-0.64	0.77	-0.84
-20	-0.34	0.94	-0.36
0	0	1.00	0
20	0.34	0.94	0.36
40	0.64	0.77	0.84
60	0.87	0.50	1.73
80	0.98	0.17	5.67
85	0.996	0.087	11.43
89	0.9998	0.0175	57.29
89.9	0.999998	0.00175	572.96

a. **As $\theta \to -90°$ and $\theta > -90°$, what value does $\sin(\theta°)$ approach?**

-1

b. **As $\theta \to -90°$ and $\theta > -90°$, what value does $\cos(\theta°)$ approach?**

0

c. **As $\theta \to -90°$ and $\theta > -90°$, how would you describe the value of $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$?**

$\tan(\theta°) \to -\infty$

d. As $\theta \to 90°$ and $\theta < 90°$, what value does $\sin(\theta°)$ approach?

1

e. As $\theta \to 90°$ and $\theta < 90°$, what value does $\cos(\theta°)$ approach?

0

f. As $\theta \to 90°$ and $\theta < 90°$, how would you describe the behavior of $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$?

$\tan(\theta°) \to \infty$

g. How can we describe the range of the tangent function?

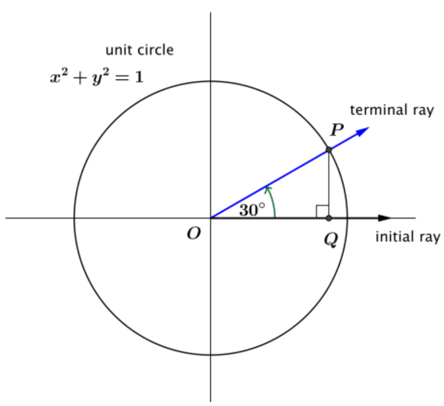
The range of the tangent function is $(-\infty, \infty)$, which is the set of all real numbers.

Example 1 (2 minutes)

Now that the domain and range of the tangent function has been established, go through a concrete example of computing the value of the tangent function at a specific value of θ; here $\theta = 30$ is used. With students, use either $\tan(\theta°) = \frac{y_\theta}{x_\theta}$ or $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ as a working definition for the tangent function, whichever seems more appropriate for a given task.

Example 1

Suppose that point P is the point on the unit circle obtained by rotating the initial ray through $30°$. Find $\tan(30°)$.



Scaffolding:

For struggling students, provide a review of the side lengths of $30°$-$60°$-$90°$ and $45°$-$45°$-$90°$ triangles.

- What is the length OQ of the horizontal leg of $\triangle OPQ$?
  - *By remembering the special triangles from Geometry, we have $OQ = \frac{\sqrt{3}}{2}$.*
- What is the length QP of the vertical leg of $\triangle OPQ$?
  - *Either by the Pythagorean theorem, or by remembering the special triangles from Geometry, we have $QP = \frac{1}{2}$.*

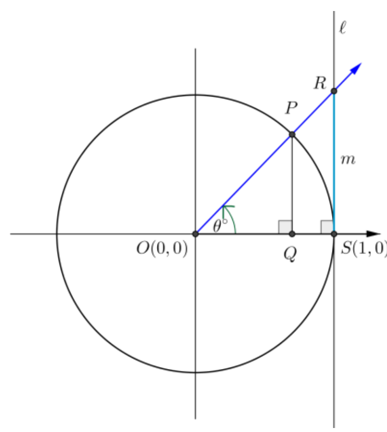
- What are the coordinates of point P?
  - $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
- What are $\cos(30°)$ and $\sin(30°)$?
  - $\cos(30°) = \frac{\sqrt{3}}{2}$, *and* $\sin(30°) = \frac{1}{2}$.
- What is $\tan(30°)$?
  - $\tan(30°) = \frac{1}{\sqrt{3}}$ *With no radicals in the denominator, this is* $\tan(30°) = \frac{\sqrt{3}}{3}$.

Exercise 2–6 (8 minutes): Why Do We Call It Tangent?

In this set of exercises, students begin to answer the question posed in the lesson's title: Why Call It Tangent? Ask students if they can see any reason to name the function $f(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ the tangent function. It is unlikely that they will have a reasonable answer.

Exercises 2–6

2. Let P be the point on the unit circle with center O that is the intersection of the terminal ray after rotation by θ degrees as shown in the diagram. Let Q be the foot of the perpendicular line from P to the x-axis, and let the line ℓ be the line perpendicular to the x-axis at $S(1,0)$. Let R be the point where the secant line OP intersects the line ℓ. Let m be the length of $\overline{RS}$.



a. Show that $m = \tan(\theta°)$.

Segment RS has length m, and side $\overline{OS}$ adjacent to $\angle ROS$ has length 1, so we use tangent:

$$\tan(\theta°) = \frac{m}{1} = m$$

Thus, $m = \tan(\theta°)$.

MP.3

b. Using a segment in the figure, make a conjecture why mathematicians named the function $f(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ the tangent function.

Mathematicians named the function tangent *because the value of the function is the length of the segment RS that is tangent to the circle.*

c. Why can you use either triangle, $\triangle POQ$ or $\triangle ROS$, to calculate the length m?

These triangles are similar by AA similarity (both are right triangles that share a common acute angle); hence, their sides are proportional.

d. Imagine that you are the mathematician who gets to name the function. (How cool would that be?) Based upon what you know about the equations of lines, what might you have named the function instead?

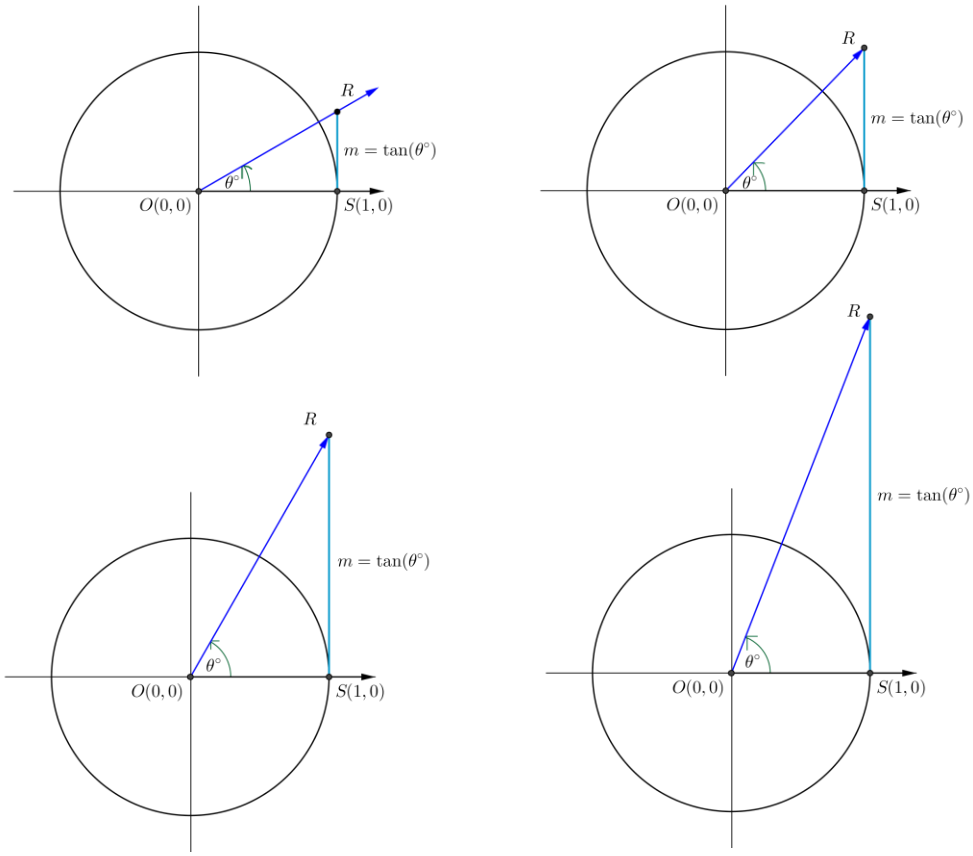
I would have called it the slope function instead because the slope of the secant line OR is also m.

Scaffolding:

- It may help students who struggle to see the diagram drawn both with and without the half-chord PQ drawn.
- For Exercise 2, part (b), students may need to be reminded that the side lengths of $\triangle POQ$ are $\sin(\theta°)$ and $\cos(\theta°)$.

MP.7 & MP.8

3. Draw four pictures similar to the diagram in Exercise 2 to illustrate what happens to the value of $\tan(\theta^\circ)$ as the rotation of the secant line through the terminal ray increases towards 90°. How does your diagram relate to the work done in Exercise 1?



As the rotation increases to 90 degrees, the value of m increases to positive infinity. Since $m = \tan(\theta^\circ)$, the value of $\tan(\theta^\circ)$ also increases to positive infinity. This is what was observed numerically in Exercise 1.

4. When the terminal ray is vertical, what is the relationship between the secant line OR and the tangent line RS? Explain why you cannot determine the measure of m in this instance. What is the value of $\tan(90^\circ)$?

The secant line and tangent line are parallel when the terminal ray is vertical. The value of m cannot be determined because parallel lines do not intersect. Therefore, no triangle is formed, and triangle trigonometry does not apply. The tangent function is undefined at $\theta = 90$.

5. When the terminal ray is horizontal, what is the relationship between the secant line OR and the x-axis? Explain what happens to the value of m in this instance. What is the value of $\tan(0^\circ)$?

When the secant line OR is horizontal, then it coincides with the x-axis. The value of m is zero since the points R and S are the same. Then, $m = 0$, and thus $\tan(0^\circ) = 0$.

6. When the terminal ray is rotated counterclockwise about the origin by 45°, what is the relationship between the value of m and the length of $\overline{OS}$? What is the value of $\tan(45^\circ)$?

In this case, $m = OS$. Since $\overline{OS}$ is a radius of the unit circle, $m = 1$. Thus, $\tan(45^\circ) = 1$.

While debriefing this set of exercises, make sure to emphasize the following points:

- For rotations from 0 to 90 degrees, the length of the tangent segment formed by intersecting the terminal ray with the line tangent to the unit circle at $(1,0)$ is equal to $\tan(\theta°)$.
- The tangent function is undefined when $\theta = 90°$. This fact can now be related to fact that the terminal ray and the line tangent to the unit circle at $(1,0)$ will be parallel after a 90 degree rotation; thus, a tangent segment for this rotation does not exist.
- The value of the tangent function when $\theta = 0°$ is 0 because the point where the terminal ray intersects the tangent line is the point $(1,0)$, and the distance between a point and itself is 0.

Exercises 7–8 (9 minutes)

MP.7

In these exercises, students discover the relationship between $\tan(\theta°)$ and the slope of the secant line through the origin that makes an angle of θ degrees with the x-axis for rotations that place the terminal ray in the first and third quadrants. The interpretation of the tangent of θ as the slope of this secant line provides a geometric explanation why the fundamental period of the tangent function is $180°$, as opposed to the fundamental period of $360°$ for the sine and cosine functions.

Students should work in collaborative groups or with a partner on these exercises. Then as a whole group, debrief the results and provide time for students to revise what they wrote initially.

Scaffolding:

Students who are struggling to remember the sine values may be encouraged to recall the sequence $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$ as these are the values of sine at 0, 30, 45, 60, and 90 degrees.

Exercises 7–8

7. Rotate the initial ray about the origin the stated number of degrees. Draw a sketch and label the coordinates of point P where the terminal ray intersects the unit circle. What is the slope of the line containing this ray?

a. $30°$

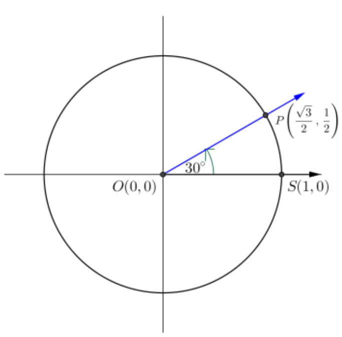
$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{1}{2} - 0}{\frac{\sqrt{3}}{2} - 0} = \frac{\sqrt{3}}{3}$$

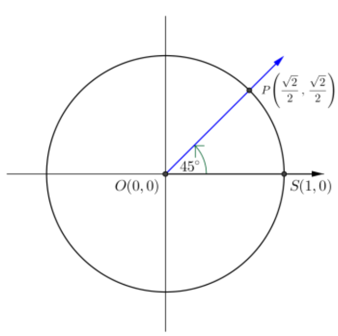
b. $45°$

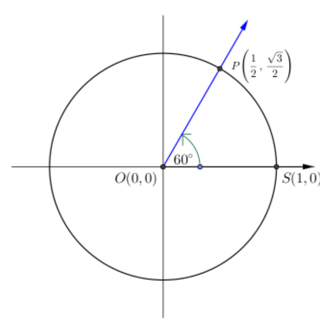
$$m = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

c. $60°$

$$m = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$







d. **Use the definition of tangent to find $\tan(30°)$, $\tan(45°)$, and $\tan(60°)$. How do your answers compare to your work in parts (a)–(c)?**

The slopes and the values of the tangent functions at each rotation were the same. That is, $\tan(30°) = \frac{\sqrt{3}}{3}$, $\tan(45°) = 1$, *and* $\tan(60°) = \sqrt{3}$.

e. **If the initial ray is rotated θ degrees about the origin, show that the slope of the line containing the terminal ray is equal to $\tan(\theta°)$. Explain your reasoning.**

The terminal ray will always intersect the unit circle at $(x_\theta, y_\theta) = (\cos(\theta°), \sin(\theta°))$ *and will always pass through the origin,* $(0, 0)$. *Thus, the slope of the line containing the terminal ray will always be given by* $m = \frac{\sin(\theta°) - 0}{\cos(\theta°) - 0} = \frac{\sin(\theta°)}{\cos(\theta°)} = \tan(\theta°)$.

f. **Now that you have shown that the value of the tangent function is equal to the slope of the terminal ray, would you prefer using the name *tangent function* or *slope function*? Why do you think we use *tangent* instead of *slope* as the name of the tangent function?**

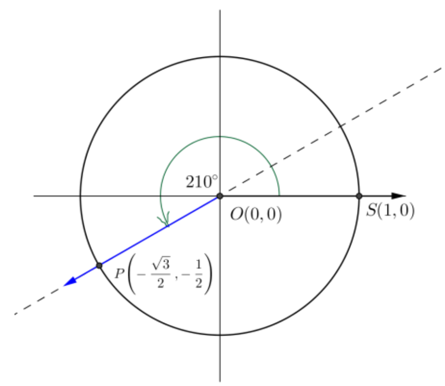
Answers may vary for the first question. Possible reasons include familiarity or comfort with one of the concepts. Tangent is probably used instead of slope due to the historical ideas of trigonometry and geometry.

8. **Rotate the initial ray about the origin the stated number of degrees. Draw a sketch and label the coordinates of point P where the terminal ray intersects the unit circle. How does your diagram in this exercise relate to the diagram in the corresponding part of Exercise 7? What is $\tan(\theta°)$ for these values of θ?**

a. **$210°$**

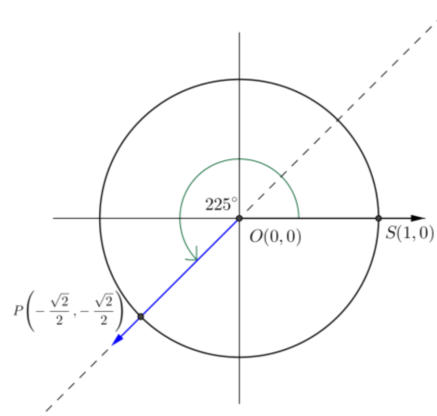
From the picture, we can see that the secant is the same line produced by a rotation by $30°$,

so $\tan(210°) = \tan(30°) = \frac{\sqrt{3}}{3}$.



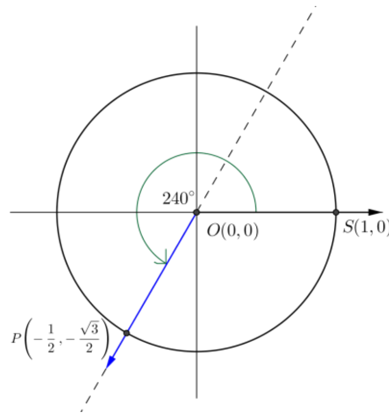
b. **$225°$**

From the picture, we can see that the secant is the same line produced by a rotation by $45°$,
so $\tan(225°) = \tan(45°) = 1$.



c. $240°$

From the picture, we can see that the secant is the same line produced by a rotation by $60°$, so $\tan(240°) = \tan(60°) = \sqrt{3}$.



d. What do the results of parts (a)–(c) suggest about the value of the tangent function after rotating an additional 180 degrees?

If the initial ray is rotated by $180 + \theta$ degrees, then the terminal ray is on the same line as the terminal ray when the initial ray is rotated by θ degrees. Since the slope of the line containing the two terminal rays is the same, the value of the tangent function for both θ and $180 + \theta$ is the same.

e. What is the period of the tangent function? Discuss with a classmate and write your conclusions.

The period is 180 since rotation by 180 degrees rotates a line to itself.

f. Use the results of Exercise 7(e) to explain why $\tan(0°) = 0$.

The slope of any horizontal line is zero.

g. Use the results of Exercise 7(e) to explain why $\tan(90°)$ is undefined.

The slope of any vertical line is undefined.

Closing (3 minutes)

In this lesson, we saw three ways to interpret the tangent function:

1. We have a working definition of tangent as $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, where $\cos(\theta°) \neq 0$.
2. Using similar triangles, we found that $\tan(\theta°) = m$, where m is the length of the line segment contained in the line ℓ tangent to the unit circle at $(1,0)$ between the point $(1,0)$ and the point of intersection of the terminal ray and line ℓ.
3. Applying the formula for slope, we see that $\tan(\theta°) = m$, where m is the slope of the secant line that contains the terminal ray of a rotation by θ degrees.

Have students summarize these interpretations of $\tan(\theta°)$ in this lesson along with the domain and range of this new function, as well as any other information they learned that they feel is important either as a class or with a partner. Use this as an opportunity to check for any gaps in understanding.

Lesson Summary

- A working definition of the tangent function is $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, where $\cos(\theta°) \neq 0$.
- The value of $\tan(\theta°)$ is the length of the line segment on the tangent line to the unit circle centered at the origin from the intersection with the unit circle and the intersection with the secant line created by the x-axis rotated θ degrees (This is why we call it tangent.)
- The value of $\tan(\theta°)$ is the slope of the line obtained by rotating the x-axis θ degrees about the origin.
- The domain of the tangent function is $\{\theta \in \mathbb{R} | \theta \neq 90 + 180k, \text{for all integers } k\}$ which is equivalent to $\{\theta \in \mathbb{R} | \cos(\theta°) \neq 0\}$.
- The range of the tangent function is all real numbers.
- The period of the tangent function is $180°$.

$\tan(0°)$	$\tan(30°)$	$\tan(45°)$	$\tan(60°)$	$\tan(90°)$
0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

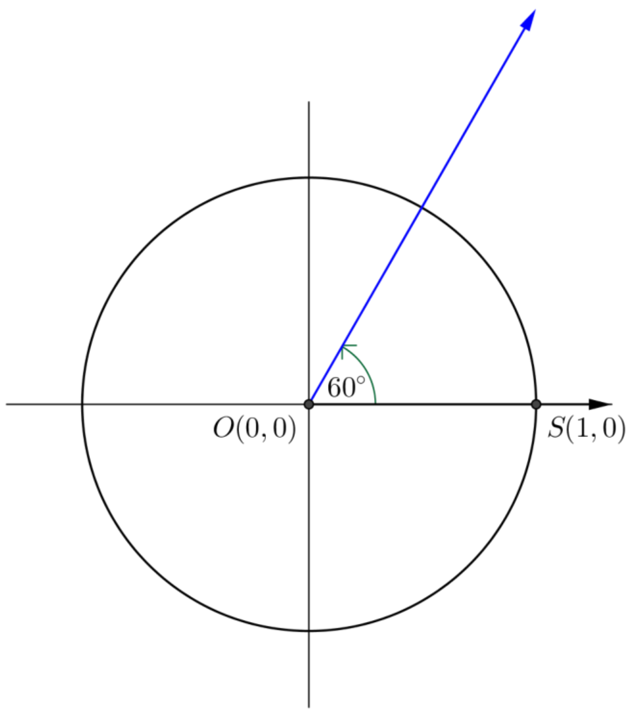
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 6: Why Call It Tangent?

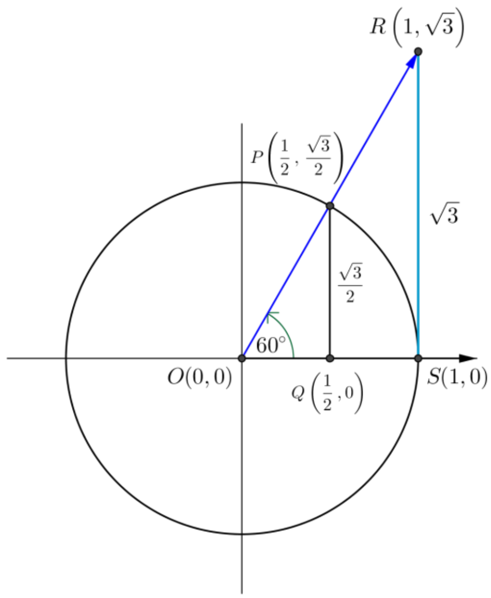
Exit Ticket

Draw and label a figure on the circle that illustrates the relationship of the trigonometric tangent function $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$ and the geometric tangent line to a circle through the point $(1,0)$ when $\theta = 60$. Explain the relationship, labeling the figure as needed.



Exit Ticket Sample Solutions

Draw and label a figure on the circle that illustrates the relationship of the trigonometric tangent function $\tan(\theta^\circ) = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$ and the geometric tangent line to a circle through the point $(1, 0)$ when $\theta = 60^\circ$. Explain the relationship, labeling the figure as needed.

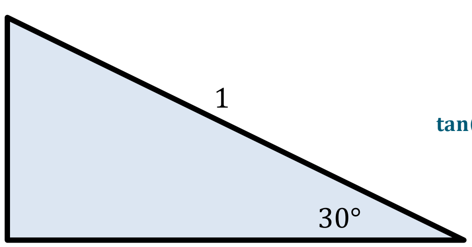


Labeling as shown, lengths are $OS = 1$, $QP = \sin(60^\circ) = \frac{\sqrt{3}}{2}$, and $OQ = \cos(60^\circ) = \frac{1}{2}$.

Then by similar triangles, we have $\frac{QP}{OQ} = \frac{SR}{OS}$; thus, $SR = \tan(60^\circ) = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$.

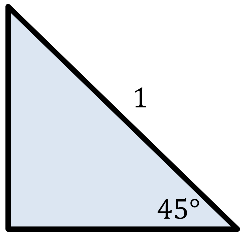
Problem Set Sample Solutions

1. **Label the missing side lengths, and find the value of $\tan(\theta^\circ)$ in the following right triangles.**
   a. $\theta = 30$



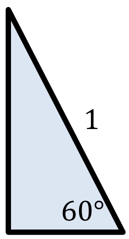
$$\tan(30^\circ) = \frac{\sin(30^\circ)}{\cos(30^\circ)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

b. $\theta = 45$



$$\tan(45°) = \frac{\sin(45°)}{\cos(45°)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

c. $\theta = 60$



$$\tan(60°) = \frac{\sin(60°)}{\cos(60°)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

2. Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get point $P(x_\theta, y_\theta)$.

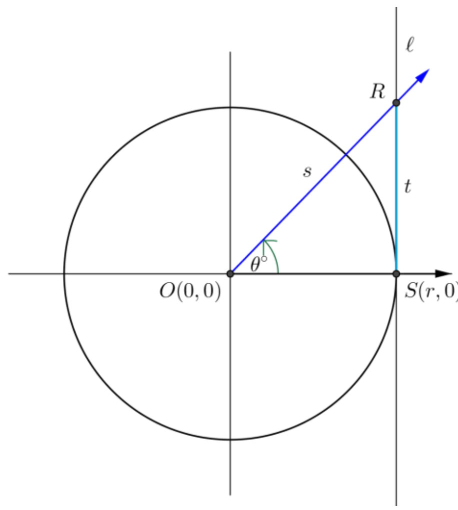
a. Complete the table by finding the slope of the line through the origin and the point P.

θ, in degrees	Slope	θ, in degrees	Slope
0	0	180	0
30	$\frac{\sqrt{3}}{3}$	210	$\frac{\sqrt{3}}{3}$
45	1	225	1
60	$\sqrt{3}$	240	$\sqrt{3}$
90	*Undefined*	270	*Undefined*
120	$-\sqrt{3}$	300	$-\sqrt{3}$
135	-1	315	-1
150	$-\frac{\sqrt{3}}{3}$	330	$-\frac{\sqrt{3}}{3}$

b. Explain how these slopes are related to the tangent function.

The slope of the line through the origin and $P(x_\theta, y_\theta)$ is equal to $\tan(\theta°)$.

3. **Consider the following diagram of a circle of radius r centered at the origin. The line ℓ is tangent to the circle at $S(r, 0)$, so ℓ is perpendicular to the x-axis.**



a. **If $r = 1$, then state the value of t in terms of one of the trigonometric functions.**

$t = \tan(\theta°)$

b. **If r is any positive value, then state the value of t in terms of one of the trigonometric functions.**

$t = r\tan(\theta°)$

For the given values of r and θ, find t.

c. $\theta = 30, r = 2$

$t = 2 \cdot \tan(30°) = \frac{2\sqrt{3}}{3}$

d. $\theta = 45, r = 2$

$t = 2 \cdot \tan(45°) = 2 \cdot 1 = 2$

e. $\theta = 60, r = 2$

$t = 2 \cdot \tan(60°) = 2 \cdot \sqrt{3} = 2\sqrt{3}$

f. $\theta = 45, r = 4$

$t = 4 \cdot \tan(45°) = 4 \cdot 1 = 4$

g. $\theta = 30, r = 3.5$

$t = 3.5 \cdot \tan(30°) = \frac{3.5\sqrt{3}}{3} = \frac{7\sqrt{3}}{6}$

h. $\theta = 0, r = 9$

$t = 9 \cdot \tan(0°) = 9 \cdot 0 = 0$

i. $\theta = 90, r = 5$

Lines OR and ℓ are distinct parallel lines when $\theta = 90$. Thus, they will never intersect, and the line segment defined by their intersection does not exist.

j. $\theta = 60, r = \sqrt{3}$

$t = \sqrt{3} \cdot \tan(60°) = \sqrt{3} \cdot \sqrt{3} = 3$

k. $\theta = 30, r = 2.1$

$t = 2.1 \cdot \tan(30°) = \frac{2.1}{\sqrt{3}} = \frac{21}{10\sqrt{3}} = \frac{7\sqrt{3}}{10}$

l. $\theta = A, r = 3$

$t = 3 \cdot \tan(A°) = 3\tan(A°)$*, for* $A \neq 90 + 180k$*, for all integers* k*.*

m. $\theta = 30, r = b$

$t = b \cdot \tan(30°) = \frac{b\sqrt{3}}{3}$

n. Knowing that $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, for $r = 1$, find the value of s in terms of one of the trigonometric functions.

Using right-triangle trigonometry, $\sin(\theta°) = \frac{t}{s} = \frac{\tan(\theta°)}{s}$.

So, $\sin(\theta°) = \frac{\tan(\theta°)}{s}$*, which tells us* $\frac{1}{\sin(\theta°)} = \frac{s}{\tan(\theta°)}$.

Thus, $s = \frac{\tan(\theta°)}{\sin(\theta°)} = \frac{\sin(\theta°)/\cos(\theta°)}{\sin(\theta°)} = \frac{\sin(\theta°)}{\cos(\theta°)} \cdot \frac{1}{\sin(\theta°)} = \frac{1}{\cos(\theta°)}$.

So, $s = \frac{1}{\cos(\theta°)}$.

4. Using what you know of the tangent function, show that $-\tan(\theta°) = \tan(-\theta°)$ for $\theta \neq 90 + 180k$, for all integers k.

The tangent function could also be called the slope function due to the fact that $\tan(\theta°)$ is the slope of the secant line passing through the origin and intersecting the tangent line perpendicular to the x-axis. If rotation of the secant line by $\theta°$ is a counterclockwise rotation, then rotation of the secant line by $-\theta°$ is a clockwise rotation. The resulting secant lines will have opposite slopes, so the tangent values will also be opposites.
Thus, $-\tan(\theta°) = \tan(-\theta°)$.

Lesson 7: Secant and the Co-Functions

Student Outcomes

- Students define the secant function and the co-functions in terms of points on the unit circle. They relate the names for these functions to the geometric relationships among lines, angles, and right triangles in a unit circle diagram.
- Students use reciprocal relationships to relate the trigonometric functions to each other and use these relationships to evaluate trigonometric functions at multiples of 30, 45, and 60 degrees.

Lesson Notes

The geometry of the unit circle and its related triangles provide a clue as to how the different reciprocal functions got their names. This lesson draws out the connections among tangent and secant lines of a circle, angle relationships, and the trigonometric functions. The names for the various trigonometric functions make more sense to students when viewed through the lens of geometric figures, providing students with an opportunity to practice MP.7. Students make sense of the domain and range of these functions and use the definitions to evaluate the trigonometric functions for rotations that are multiples of 30, 45, and 60 degrees.

The relevant vocabulary upon which this lesson is based appears below.

SECANT FUNCTION (description): The *secant function*,

$$\sec: \{x \in \mathbb{R} \mid x \neq 90 + 180k \text{ for all integers } k\} \to \mathbb{R}$$

can be defined as follows: Let θ be any real number such that $\theta \neq 90 + 180k$ for all integers k. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\sec(\theta°)$ is $\frac{1}{x_\theta}$.

COSECANT FUNCTION (description): The *cosecant function*,

$$\csc: \{x \in \mathbb{R} \mid x \neq 180k \text{ for all integers } k\} \to \mathbb{R}$$

can be defined as follows: Let θ be any real number such that $\theta \neq 180k$ for all integers k. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\csc(\theta°)$ is $\frac{1}{y_\theta}$.

COTANGENT FUNCTION (description): The *cotangent function*,

$$\cot: \{x \in \mathbb{R} \mid x \neq 180k \text{ for all integers } k\} \to \mathbb{R}$$

can be defined as follows: Let θ be any real number such that $\theta \neq 180k$ for all integers k. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\cot(\theta°)$ is $\frac{x_\theta}{y_\theta}$.

Classwork

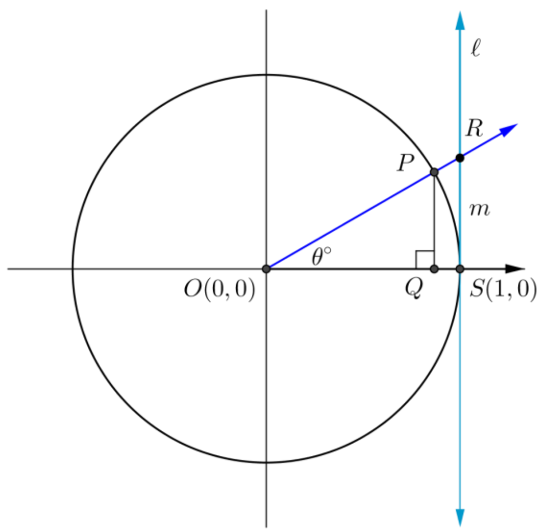
Opening Exercise (5 minutes)

Give students a short time to work independently on this Opening Exercise (about 2 minutes). Lead a whole-class discussion afterward to reinforce the reasons why the different segments have the given measures. If students have difficulty naming these segments in terms of trigonometric functions they have already studied, remind them of the conclusions of the previous few lessons.

Opening Exercise

Find the length of each segment below in terms of the value of a trigonometric function.

$OQ =$ ____________ $PQ =$ ____________ $RS =$ ____________



Debrief this exercise with a short discussion. For the purposes of this section, limit the values of θ to be between 0 and 90. Make sure that each student has labeled the proper line segments on his paper as $\sin(\theta°)$, $\cos(\theta°)$, $\tan(\theta°)$, and $\sec(\theta°)$ before moving on to Example 1.

- Why is $PQ = \sin(\theta°)$? Why is $OQ = \cos(\theta°)$?
  - *The values of the sine and cosine functions correspond to the y- and x-coordinates of a point on the unit circle where the terminal ray intersects the circle after a rotation of θ degrees about the origin.*
- Why is $RS = \tan(\theta°)$?
  - *In Lesson 5, we used similar triangles to show that $\overline{RS}$ had length* $\dfrac{PQ}{OQ} = \dfrac{\sin(\theta°)}{\cos(\theta°)}$.

    This quotient is $\tan(\theta°)$.

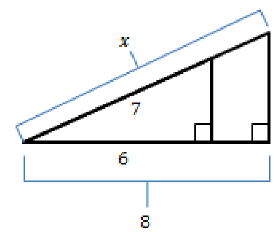
Since there are ways to calculate lengths for nearly every line segment in this diagram using the length of the radius, or the cosine, sine, or tangent functions, it makes sense to find the length of $\overline{OR}$, the line segment on the terminal ray that intersects the tangent line.

- What do you call a line that intersects a circle at more than one point?
  - *It is called a secant line.*
- In Lesson 6, we saw that $RS = \tan(\theta°)$, where $\overline{RS}$ lies on the line tangent to the unit circle at $(1,0)$, which helped to explain how this trigonometric function got its name. Let's introduce a new function $\sec(\theta°)$, the secant of θ, to be the length of $\overline{OR}$ since this segment is on the secant line that contains the terminal ray. Then the *secant* of θ is $\sec(\theta°) = OR$.

Scaffolding:

- Have students use different-colored highlighters or pencils to mark proportional segments.

  Consider writing out the proportional relationships using the segment names first. For example, $\frac{OP}{OQ} = \frac{OR}{OS}$.
- Provide a numeric example for students to work first.



- For more advanced students, skip the Opening Exercise and start by stating that a new function, $\sec(\theta°) = OR$, is being introduced since $\overline{OR}$ is on the secant line that passes through the center of the circle.

Example 1 (5 minutes)

Direct students to label the appropriate segments on their papers if they have not already done so. Reproduce the Opening Exercise diagram on chart paper, and label the segments on the diagram in a different color than the rest of the diagram so they are easy to see. Refer to this chart often in the next section of the lesson.

Example 1

Use similar triangles to find the value of $\sec(\theta°)$ in terms of one other trigonometric function.

Approach 1:

By similar triangles, $\frac{OR}{OS} = \frac{OP}{OQ}$, *so* $\frac{\sec(\theta°)}{1} = \frac{1}{\cos(\theta°)}$; *thus,* $\sec(\theta°) = \frac{1}{\cos(\theta°)}$.

Approach 2:

By similar triangles $\frac{OR}{OS} = \frac{OP}{OQ}$, *so* $\frac{\sec(\theta°)}{1} = \frac{\tan(\theta°)}{\sin(\theta°)} = \frac{\sin(\theta°)/\cos(\theta°)}{\sin(\theta°)} = \frac{\sin(\theta°)}{\cos(\theta°)} \cdot \frac{1}{\sin(\theta°)} = \frac{1}{\cos(\theta°)}$; *thus,* $\sec(\theta°) = \frac{1}{\cos(\theta°)}$.

Exercise 1 (5 minutes)

Following the same technique as in the previous lesson with the tangent function, use this working definition of the secant function to extend it outside of the first quadrant. Present the definition of secant, and then ask students to answer the following questions with a partner or in writing. Circulate around the classroom to informally assess understanding and provide guidance. Lead a whole-class discussion based on these questions after giving individuals or partners a few minutes to record their thoughts. Encourage students to revise what they wrote as the discussion progresses. Start a bulleted list of the main points of this discussion on the board. Encourage students to draw a picture representing both the circle description of the secant function and its relationship to $\cos(\theta°)$ to assist them in answering the question. A series of guided questions follows that help in scaffolding this discussion.

Exercise 1

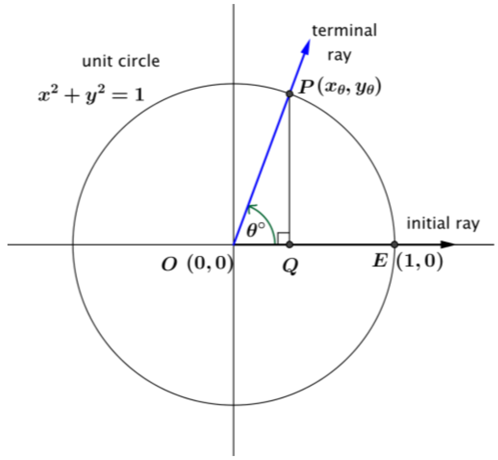
The definition of the secant function is offered below. Answer the questions to better understand this definition and the domain and range of this function. Be prepared to discuss your responses with others in your class.

Let θ be any real number.

In the Cartesian plane, rotate the nonnegative x-axis by θ degrees about the origin. Intersect this new ray with the unit circle to get a point (x_θ, y_θ).

If $x_\theta \neq 0$, then the value of $\sec(\theta°)$ is $\frac{1}{x_\theta}$. Otherwise, $\sec(\theta°)$ is undefined.

In terms of the cosine function, $\sec(\theta°) = \frac{1}{\cos(\theta°)}$ for $\cos(\theta°) \neq 0$.



MP.3

a. **What is the domain of the secant function?**

The domain of the secant function is all real numbers θ such that $\theta \neq 90 + 180k$, for all integers k.

b. **The domains of the secant and tangent functions are the same. Why?**

Both the tangent and secant functions can be written as rational expressions with $\cos(\theta°)$ in the denominator. That is, $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, and $\sec(\theta°) = \frac{1}{\cos(\theta°)}$. Since the restricted values in their domains occur when the denominator is zero, it makes sense that they have the same domain.

c. **What is the range of the secant function? How is this range related to the range of the cosine function?**

Since $\sec(\theta°) = \frac{1}{\cos(\theta°)}$, and $-1 \leq \cos(\theta°) \leq 1$, we see that either $\sec(\theta°) \geq 1$, or $\sec(\theta°) \leq -1$.

Thus, the range of the secant function is $(-\infty, -1] \cup [1, \infty)$.

d. **Is the secant function a periodic function? If so, what is its period?**

Yes, its period is 360.

Discussion (5 minutes)

Use these questions to scaffold the discussion to debrief the preceding exercise.

- What are the values of $\sec(\theta°)$ when the terminal ray is horizontal? When the terminal ray is vertical?
  - *When the terminal ray is horizontal, it will coincide with the positive or negative x-axis, and $\sec(\theta°) = \frac{1}{1} = 1$, or $\sec(\theta°) = \frac{1}{-1} = -1$.*
  - *When the terminal ray is vertical, the x-coordinate of point P is zero, so $\sec(\theta°) = \frac{1}{0}$, which is undefined. If we use the geometric interpretation of $\sec(\theta°)$ as the length of the segment from the origin to the intersection of the terminal ray and the tangent line to the circle at $(1,0)$, then $\sec(\theta°)$ is undefined because the secant line containing the terminal ray and the tangent line will be parallel and will not intersect.*

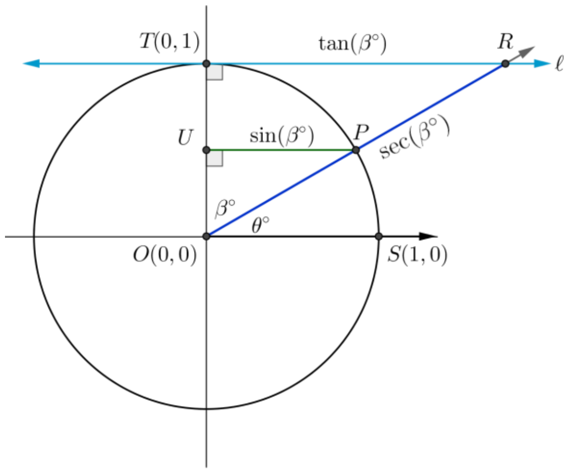
- Name several values of θ for which $\sec(\theta°)$ is undefined. Explain your reasoning.
  - *The secant of θ is undefined when $\cos(\theta°) = 0$ or when the terminal ray intersects the unit circle at the y-axis. Some values of θ that meet these conditions include $90°$, $270°$, and $450°$.*
- What is the domain of the secant function?
  - Answers will vary but should be equivalent to all real numbers except $90 + 180k$, where k is an integer. Students may also use set-builder notation such as $\{\theta \in \mathbb{R} | \cos(\theta°) \neq 0\}$.
- The domains of the tangent and secant functions are the same. Why?
  - *Approach 1 (circle approach):*
    - *The tangent and secant segments are defined based on their intersection, so at times when they are parallel, these segments do not exist.*
  - *Approach 2 (working definition approach):*
    - *Both the tangent and secant functions are defined as rational expressions with $\cos(\theta)$ in the denominator. That is, $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, and $\sec(\theta°) = \frac{1}{\cos(\theta°)}$. Since the restricted values in their domains occur when the denominator is zero, it makes sense that they have the same domain.*
- How do the values of the secant and cosine functions vary with each other? As $\cos(\theta°)$ gets larger, what happens to the value of $\sec(\theta°)$? As $\cos(\theta°)$ gets smaller but stays positive, what happens to the value of $\sec(\theta°)$? What about when $\cos(\theta°) < 0$?
  - *We know that $\sec(\theta°)$ and $\cos(\theta°)$ are reciprocals of each other, so as the magnitude of one gets larger, the magnitude of the other gets smaller and vice versa. As $\cos(\theta°)$ increases, $\sec(\theta°)$ gets closer to 1. As $\cos(\theta°)$ decreases but stays positive, $\sec(\theta°)$ increases without bound. When $\cos(\theta°)$ is negative, as $\cos(\theta°)$ increases but stays negative, $\sec(\theta°)$ decreases without bound. As $\cos(\theta°)$ decreases, $\sec(\theta°)$ gets closer to -1.*
- What is the smallest positive value of $\sec(\theta°)$? Where does this occur?
  - *The smallest positive value of $\sec(\theta°)$ is 1 and occurs when $\cos(\theta°) = 1$, that is, when $\theta = 360k$, for k an integer.*
- What is the largest negative value of $\sec(\theta°)$? Where does this occur?
  - *The largest negative value of $\sec(\theta°) = -1$ and occurs when $\cos(\theta°) = -1$; that is, when $\theta = 180 + 360k$, for k an integer.*
- What is the range of the secant function?
  - *All real numbers outside of the interval $(-1,1)$, including -1 and 1.*
- Is the secant function a periodic function? If so, what is its period?
  - *The secant function is periodic, and the period is $360°$.*

Exercise 2 (5 minutes)

These questions get students thinking about the origin of the names of the co-functions. They start with the diagram from the Opening Exercise and ask students how the diagram below compares to it. Then, the point of Example 2 is to introduce the sine, secant, and tangent ratios of the complement to θ and justify why we name these functions cosine, cosecant, and cotangent.

Exercise 2

In the diagram, the horizontal blue line is tangent to the unit circle at $(0, 1)$.



a. **How does this diagram compare to the one given in the Opening Exercise?**

It is basically the same diagram with different angles and lengths marked. We can consider a rotation through β degrees measured clockwise from the y-axis instead of a rotation through θ degrees measured counterclockwise from the x-axis. It is essentially a reflection of the diagram from the Opening Exercise across the diagonal line with equation $y = x$.

b. **What is the relationship between β and θ?**

Angles TOP and SOP are complements, so $\beta + \theta = 90$.

MP.7

c. **Which segment in the figure has length $\sin(\theta°)$? Which segment has length $\cos(\theta°)$?**

$OU = \sin(\theta°)$, and $UP = \cos(\theta°)$.

d. **Which segment in the figure has length $\sin(\beta°)$? Which segment has length $\cos(\beta°)$?**

$UP = \sin(\beta°)$, and $OU = \cos(\theta°)$.

e. **How can you write $\sin(\theta°)$ and $\cos(\theta°)$ in terms of the trigonometric functions of β?**

$\sin(\theta°) = \cos(\beta°)$, and $\cos(\theta°) = \sin(\beta°)$.

Scaffolding:

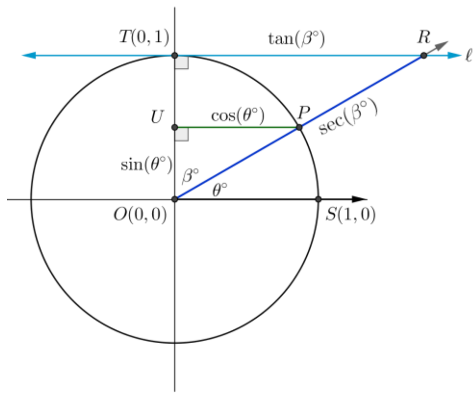
- Students struggling to see the complementary relationships among the trigonometric functions may benefit from drawing the unit circle on a transparency or patty paper and reflecting across the diagonal line $y = x$ so that the x-axis and y-axis are interchanged.
- Students may need to be shown where $\sin(\theta°)$ is on the diagram as well.

Briefly review the solutions to these exercises before moving on. Reinforce the result of part (e) that the cosine of an acute angle is the same as the sine of its complement. Ask students to consider whether the trigonometric ratios of the complement of an angle might be related to the original angle in a similar fashion. Record student responses to the exercises and any other predictions or thoughts on another sheet of chart paper.

Example 2 (5 minutes)

Use similar triangles to show that if $\theta + \beta = 90$, then $\sec(\beta°) = \frac{1}{\sin(\theta°)}$ and $\tan(\beta°) = \frac{1}{\tan(\theta°)}$. Depending on students' level, provide additional scaffolding or just have groups work independently on this. Circulate around the classroom as they work.

Example 2



The horizontal blue line is tangent to the circle at $(0, 1)$.

a. **If two angles are complements with measures β and θ as shown in the diagram, use similar triangles to show that $\sec(\beta°) = \frac{1}{\sin(\theta°)}$.**

Consider triangles $\triangle OPU$ and $\triangle ORT$. Then $\frac{OR}{OP} = \frac{TR}{UP}$, so $\frac{\sec(\beta°)}{1} = \frac{\tan(\beta°)}{\cos(\theta°)}$. Since $\cos(\theta°) = \sin(\beta°)$ and $\sin(\theta°) = \cos(\beta°)$, we can write

$$\sec(\beta°) = \frac{\left(\frac{\sin(\beta°)}{\cos(\beta°)}\right)}{\cos(\theta°)}$$
$$= \frac{\left(\frac{\cos(\theta°)}{\sin(\theta°)}\right)}{\cos(\theta°)}$$
$$= \frac{1}{\sin(\theta°)}.$$

b. **If two angles are complements with measures β and θ as shown in the diagram, use similar triangles to show that $\tan(\beta°) = \frac{1}{\tan(\theta°)}$.**

Using triangles $\triangle OPU$ and $\triangle ORT$, we have $\frac{RT}{OT} = \frac{UP}{OU}$. Then $\frac{\tan(\beta°)}{1} = \frac{\cos(\theta°)}{\sin(\theta°)}$. Since $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$, then $\frac{1}{\tan(\theta°)} = \frac{\cos(\theta°)}{\sin(\theta°)}$; so, $\tan(\beta°) = \frac{1}{\tan(\theta°)}$.

Discussion (7 minutes)

The reciprocal functions $\frac{1}{\sin(\theta^\circ)}$ and $\frac{1}{\tan(\theta^\circ)}$ are called the cosecant and cotangent functions. The cosine function has already been defined in terms of a point on the unit circle. The same can be done with the cosecant and cotangent functions. All three functions introduced in this lesson are defined below.

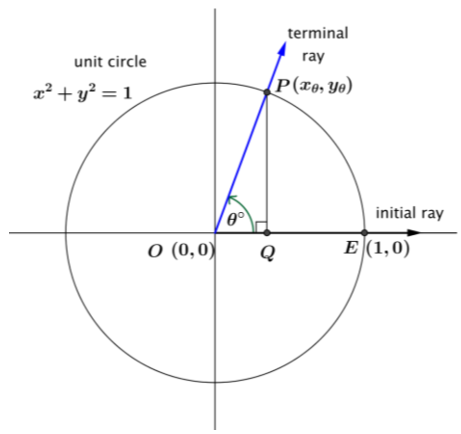
For each of the reciprocal functions, use the facts that $\sec(\theta^\circ) = \frac{1}{\cos(\theta^\circ)}$, $\csc(\theta^\circ) = \frac{1}{\sin(\theta^\circ)}$, and $\cot(\theta^\circ) = \frac{1}{\tan(\theta^\circ)}$ to extend the definitions of these functions beyond the first quadrant and to help with the following ideas. Have students work in pairs to answer the following questions before sharing their conclusions with the class.

> *Scaffolding:*
> To help students keep track of the reciprocal definitions, mention that *every trigonometric function has a co-function*.

Discussion

Definitions of the cosecant and cotangent functions are offered below. Answer the questions to better understand the definitions and the domains and ranges of these functions. Be prepared to discuss your responses with others in your class.

> **Let θ be any real number such that $\theta \neq 180k$, for all integers k.**
>
> **In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ).**
>
> **The value of $\csc(\theta^\circ)$ is $\frac{1}{y_\theta}$.**
>
> **The value of $\cot(\theta^\circ)$ is $\frac{x_\theta}{y_\theta}$.**



The secant, cosecant, and cotangent functions are often referred to as reciprocal functions. Why do you think these functions are so named?

These functions are the reciprocals of the three main trigonometric functions:

$$\sec(\theta^\circ) = \frac{1}{\cos(\theta^\circ)},\ \csc(\theta^\circ) = \frac{1}{\sin(\theta^\circ)},\ \text{and } \cot(\theta^\circ) = \frac{1}{\tan(\theta^\circ)}.$$

Why are the domains of these functions restricted?

We restrict the domain to prevent division by zero.

We restrict the domain because the geometric shapes defining the functions must make sense (i.e., based on the intersection of distinct parallel lines).

The domains of the cosecant and cotangent functions are the same. Why?

Both the cosecant and cotangent functions are equal to rational expressions that have $\sin(\theta^\circ)$ in the denominator.

What is the range of the cosecant function? How is this range related to the range of the sine function?

The sine function has range $[-1, 1]$, so the cosecant function has range $(-\infty, -1] \cup [1, \infty)$. The two ranges only intersect at -1 and at 1.

What is the range of the cotangent function? How is this range related to the range of the tangent function?

The range of the cotangent function is all real numbers. This is the same as the range of the tangent function.

- Why are the secant, cosecant, and cotangent functions called reciprocal functions?
  - *Each one is a reciprocal of cosine, sine, or tangent according to their definitions. For example,* $\csc(\theta°)$ *is equal to* $\frac{1}{y_\theta}$*, the reciprocal of* $\sin(\theta°)$*.*
- Which two of the reciprocal functions share the same domain? Why? What is their domain?
  - The cosecant and cotangent functions *share the same domain. Both functions can be written as a rational expression with* $\sin(\theta°)$ *in the denominator, so whenever* $\sin(\theta°) = 0$*, they are undefined. The domain is all real numbers except* θ *such that* $\sin(\theta°) = 0$*. More specifically, the domain is* $\{\theta \in \mathbb{R} | \theta \neq 180k, \text{for all integers } k\}$*.*
- What is the smallest positive value of the cosecant function?
  - *As with the secant function, the smallest positive value of the cosecant function is* 1*. This is because the cosecant function is the reciprocal of the sine function, which has a maximum value of* 1*. Also, the secant and cosecant functions have values based on the length of line segments from the center of the circle to the intersection with the tangent line. Thus, when positive, they must always be greater than or equal to the radius of the circle.*
- What is the greatest negative value of the cosecant function?
  - *The greatest negative value of the cosecant function is* -1*.*
- What is the range of the cosecant function? What other trigonometric function has this range?
  - *The range of the cosecant function is all real numbers except between* -1 *and* 1*; that is, the range of the cosecant function is* $(-\infty, -1] \cup [1, \infty)$*, which is the same as the range of the secant function.*
- Can $\sec(\theta°)$ or $\csc(\theta°)$ be a number between 0 and 1? Can $\cot(\theta°)$? Explain why or why not.
  - *No,* $\sec(\theta°)$ *and* $\csc(\theta°)$ *cannot be between* 0 *and* 1*, but* $\cot(\theta°)$ *can. Both the secant and cosecant functions are reciprocals of functions that range from* -1 *to* 1*, while the cotangent function is the reciprocal of a function that ranges across all real numbers. Whenever* $\tan(\theta°) > 1$*,* $0 < \cot(\theta°) < 1$*.*
- What is the value of $\cot(90°)$?
  - $\cot(90°) = \frac{\cos(90°)}{\sin(90°)} = \frac{0}{1} = 0.$
- How does the range of the cotangent function compare to the range of the tangent function? Why?
  - *The ranges of the tangent and cotangent functions are the same. When* $\tan(\theta°)$ *is close to* 0*,* $\cot(\theta°)$ *is far from* 0*, either positive or negative depending on the sign of* $\tan(\theta°)$*. When* $\tan(\theta°)$ *is far from* 0*, then* $\cot(\theta°)$ *is close to* 0*. Finally, when one function is undefined, the other is* 0*.*

Closing (4 minutes)

- The secant and cosecant functions are reciprocals of the cosine and sine functions, respectively. The tangent and cotangent functions are also reciprocals of each other. It is helpful to draw the circle diagram used to define the tangent and secant functions to ensure mistakes are not made with the relationships derived from similar triangles.

- This is a good time to summarize all six trigonometric functions. With a partner, in writing, or as a class, have students summarize the definitions for sine, cosine, tangent, and the three reciprocal functions along with their domains. Use this as an opportunity to check for any gaps in understanding. Use the following summary as a model:

Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). Then:

Function	Value	For any θ such that …	Formula
Sine	y_θ	θ is a real number	
Cosine	x_θ	θ is a real number	
Tangent	$\frac{y_\theta}{x_\theta}$	$\theta \neq 90 + 180k$, for all integers k	$\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$
Secant	$\frac{1}{x_\theta}$	$\theta \neq 90 + 180k$, for all integers k	$\sec(\theta°) = \frac{1}{\cos(\theta°)}$
Cosecant	$\frac{1}{y_\theta}$	$\theta \neq 180k$, for all integers k	$\csc(\theta°) = \frac{1}{\sin(\theta°)}$
Cotangent	$\frac{x_\theta}{y_\theta}$	$\theta \neq 180k$, for all integers k	$\cot(\theta°) = \frac{\cos(\theta°)}{\sin(\theta°)}$

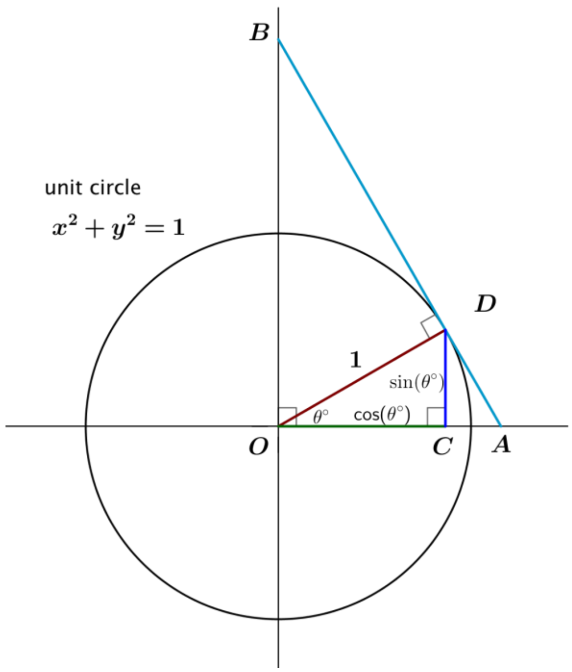
Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 7: Secant and the Co-Functions

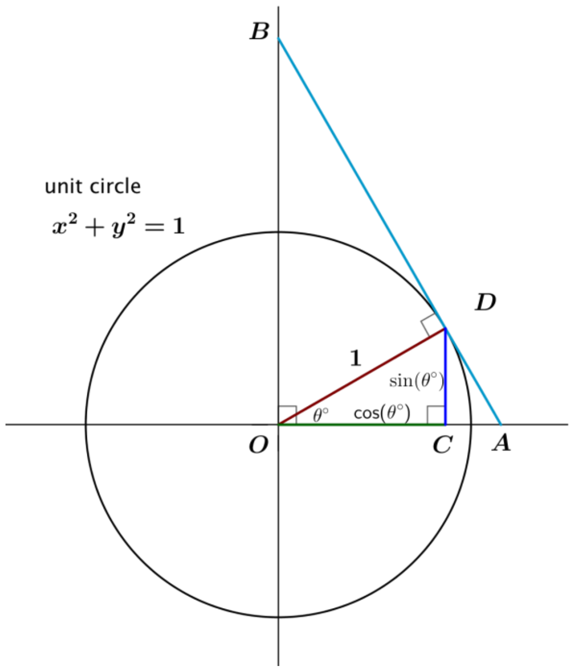
Exit Ticket

Consider the following diagram, where segment AB is tangent to the circle at D. Right triangles BAO, BOD, OAD, and ODC are similar. Identify each length AD, OA, OB, and BD as one of the following: $\tan(\theta°)$, $\cot(\theta°)$, $\sec(\theta°)$, and $\csc(\theta°)$.



Exit Ticket Sample Solutions

Consider the following diagram, where segment AB is tangent to the circle at D. Right triangles BAO, BOD, OAD, and ODC are similar. Identify each length AD, OA, OB, and BD as one of the following: $\tan(\theta°)$, $\cot(\theta°)$, $\sec(\theta°)$, and $\csc(\theta°)$.



Since $\triangle ODC \sim \triangle ODA$, $\frac{AD}{OD} = \frac{CD}{OC}$ so $\frac{AD}{1} = \frac{\sin(\theta°)}{\cos(\theta°)}$ and thus $AD = \tan(\theta°)$.

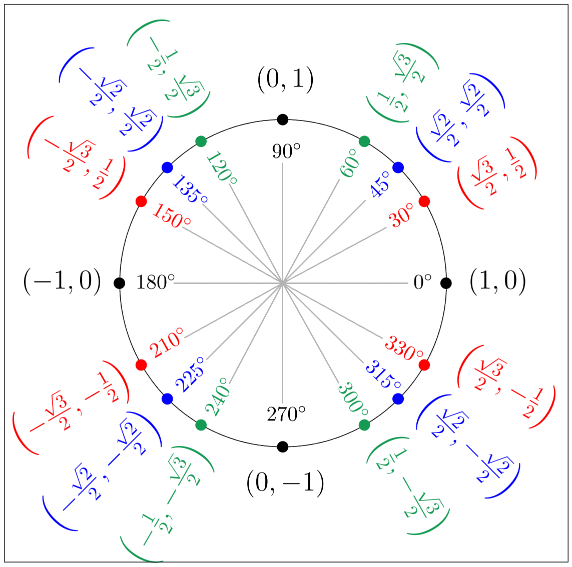
Since $\triangle ODC \sim \triangle ODA$, $\frac{OA}{OD} = \frac{OD}{OC}$ so $\frac{OA}{1} = \frac{1}{\cos(\theta°)}$ and thus $OA = \sec(\theta°)$.

Since $\triangle ODC \sim \triangle BAO$, $\frac{OB}{OA} = \frac{OC}{CD}$ so $\frac{OB}{\sec(\theta°)} = \frac{\cos(\theta°)}{\sin(\theta°)}$ and thus $OA = \sec(\theta°)\frac{\cos(\theta°)}{\sin(\theta°)} = \frac{1}{\sin(\theta°)} = \csc(\theta°)$.

Since $\triangle ODC \sim \triangle BOD$, $\frac{BD}{OD} = \frac{OC}{CD}$ so $\frac{BD}{1} = \frac{\cos(\theta°)}{\sin(\theta°)}$ and thus $BD = \cot(\theta°)$.

Problem Set Sample Solutions

1. Use the reciprocal interpretations of $\sec(\theta°)$, $\csc(\theta°)$, and $\cot(\theta°)$ and the unit circle provided to complete the table.



θ, in degrees	$\sec(\theta°)$	$\csc(\theta°)$	$\cot(\theta°)$
0	1	*Undefined*	*Undefined*
30	$\frac{2\sqrt{3}}{3}$	2	$\sqrt{3}$
45	$\sqrt{2}$	$\sqrt{2}$	1
60	2	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$
90	*Undefined*	1	0
120	-2	$\frac{2\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$
180	-1	*Undefined*	*Undefined*
225	$-\sqrt{2}$	$-\sqrt{2}$	1
240	-2	$-\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$
270	*Undefined*	-1	0
315	$\sqrt{2}$	$-\sqrt{2}$	-1
330	$\frac{2\sqrt{3}}{3}$	-2	$-\sqrt{3}$

2. Find the following values from the information given.

a. $\sec(\theta°)$; $\cos(\theta°) = 0.3$

$\sec(\theta°) = \frac{1}{0.3} = \frac{1}{3/10} = \frac{10}{3}$

b. $\csc(\theta°)$; $\sin(\theta°) = -0.05$

$\csc(\theta°) = \frac{1}{-0.05} = \frac{1}{-1/20} = -20$

c. $\cot(\theta°)$; $\tan(\theta°) = 1000$

$\cot(\theta°) = \frac{1}{\tan(\theta°)} = \frac{1}{1000}$

d. $\sec(\theta°)$; $\cos(\theta°) = -0.9$

$\sec(\theta°) = \frac{1}{-0.9} = \frac{1}{-9/10} = -\frac{10}{9}$

e. $\csc(\theta°)$; $\sin(\theta°) = 0$

$\csc(\theta°) = \frac{1}{\sin(\theta°)} = \frac{1}{0}$, *but this is undefined.*

f. $\cot(\theta°)$; $\tan(\theta°) = -0.0005$

$\cot(\theta°) = \frac{1}{-0.0005} = \frac{1}{-5/10000} = -\frac{10000}{5} = -2000$

3. Choose three θ values from the table in Problem 1 for which $\sec(\theta°)$, $\csc(\theta°)$, and $\tan(\theta°)$ are defined and not zero. Show that for these values of θ, $\frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°)$.

For $\theta = 30°$, $\tan(\theta°) = \frac{\sqrt{3}}{3}$, *and* $\frac{\sec(\theta°)}{\csc(\theta°)} = \frac{2\sqrt{3}/3}{2} = \frac{\sqrt{3}}{3}$. *Thus, for* $\theta = 30$, $\frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°)$.

For $\theta = 45°$, $\tan(\theta°) = 1$, *and* $\frac{\sec(\theta°)}{\csc(\theta°)} = \frac{\sqrt{2}}{\sqrt{2}} = 1$. *Thus, for* $\theta = 45$, $\frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°)$.

For $\theta = 60°$, $\tan(\theta°) = \sqrt{3}$, *and* $\frac{\sec(\theta°)}{\csc(\theta°)} = \frac{2}{\frac{2\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$. *Thus, for* $\theta = 60$, $\frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°)$.

4. Find the value of $\sec(\theta°)\cos(\theta°)$ for the following values of θ.

a. $\theta = 120$

We know that $\cos(120°) = -\frac{1}{2}$, *so* $\sec(120°) = -2$, *and then* $\sec(120°)\cos(120°) = 1$.

b. $\theta = 225$

We know that $\cos(225°) = -\frac{\sqrt{2}}{2}$, *so* $\sec(225°) = -\sqrt{2}$, *and then* $\sec(225°)\cos(225°) = 1$.

c. $\theta = 330$

We know that $\cos(330°) = \frac{\sqrt{3}}{2}$, *so* $\sec(330°) = \frac{2}{\sqrt{3}}$, *and then* $\sec(330°)\cos(330°) = 1$.

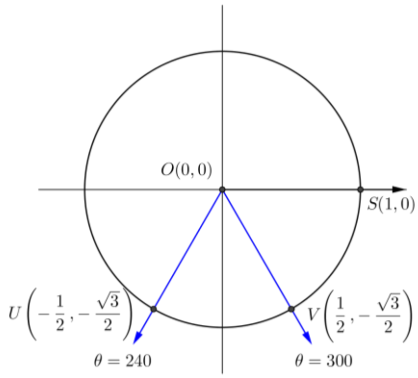
d. Explain the reasons for the pattern you see in your responses to parts (a)–(c).

If $\cos(\theta°) \neq 0$, *then* $\sec(\theta°) = \frac{1}{\cos(\theta°)}$, *so we know that* $\sec(\theta°)\cos(\theta°) = \frac{1}{\cos(\theta°)} \cdot \cos(\theta°) = 1$.

5. Draw a diagram representing the two values of θ between 0 and 360 so that $\sin(\theta°) = -\frac{\sqrt{3}}{2}$. Find the values of $\tan(\theta°)$, $\sec(\theta°)$, and $\csc(\theta°)$ for each value of θ.

   $\tan(240°) = \sqrt{3}$, $\sec(240°) = -2$, and $\csc(240°) = -\frac{2\sqrt{3}}{3}$.

   $\tan(300°) = -\sqrt{3}$, $\sec(300°) = 2$, and $\csc(300°) = -\frac{2\sqrt{3}}{3}$.



6. Find the value of $(\sec(\theta°))^2 - (\tan(\theta°))^2$ when $\theta = 225$.

   $(\sec(225°))^2 - (\tan(225°))^2 = (\sqrt{2})^2 - 1^2 = 1$

7. Find the value of $(\csc(\theta°))^2 - (\cot(\theta°))^2$ when $\theta = 330$.

   $(\csc(\theta°))^2 - (\cot(\theta°))^2 = (-2)^2 - (-\sqrt{3})^2 = 4 - 3 = 1$

Extension:

8. Using the formulas $\sec(\theta°) = \frac{1}{\cos(\theta°)}$, $\csc(\theta°) = \frac{1}{\sin(\theta°)}$, and $\cot(\theta°) = \frac{1}{\tan(\theta°)}$, show that $\frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°)$, where these functions are defined and not zero.

   $$\frac{\sec(\theta°)}{\csc(\theta°)} = \frac{\frac{1}{\cos(\theta°)}}{\frac{1}{\sin(\theta°)}} = \frac{1}{\cos(\theta°)} \cdot \sin(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)} = \tan(\theta°)$$

9. Tara showed that $\frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°)$, for values of θ for which the functions are defined and $\csc(\theta°) \neq 0$, and then concluded that $\sec(\theta°) = \sin(\theta°)$ and $\csc(\theta°) = \cos(\theta°)$. Explain what is wrong with her reasoning.

   Just because $\frac{A}{B} = \frac{C}{D}$, this does not mean that $A = C$ and $B = D$. It only means that they are equivalent fractions. For instance, $\frac{3}{4} = \frac{9}{12}$. Substituting any value of θ verifies that Tara is incorrect. Also, since the only values of θ in the range of both secant and sine are -1 and 1, we would have either $\sec(\theta°) = \sin(\theta°) = -1$ or $\sec(\theta°) = \sin(\theta°) = 1$. Since the value of the sine function is either -1 or 1, the value of the cosine function would be zero. Since $\sec(\theta°) = \frac{1}{\cos(\theta°)}$, we would have $\cos(\theta°) = 1$ or $\cos(\theta°) = -1$. Since $\cos(\theta°)$ cannot be simultaneously zero and nonzero, it is impossible to have $\sec(\theta°) = \sin(\theta°)$ and $\csc(\theta°) = \cos(\theta°)$.

10. From Lesson 6, Ren remembered that the tangent function is odd, meaning that $-\tan(\theta°) = \tan(-\theta°)$ for all θ in the domain of the tangent function. He concluded because of the relationship between the secant function, cosecant function, and tangent function developed in Problem 9, it is impossible for both the secant and the cosecant functions to be odd. Explain why he is correct.

If we assume that both the secant and cosecant functions are odd, then the tangent function could not be odd. That is, we would get

$$\tan(-\theta°) = \frac{\sec(-\theta°)}{\csc(-\theta°)} = \frac{-\sec(\theta°)}{-\csc(\theta°)} = \frac{\sec(\theta°)}{\csc(\theta°)} = \tan(\theta°),$$

but that would contradict $-\tan(\theta°) = \tan(\theta°)$. Thus, it is impossible for both the secant and cosecant functions to be odd.

Lesson 8: Graphing the Sine and Cosine Functions

Student Outcomes

- Students graph the sine and cosine functions and analyze the shape of these curves.
- For the sine and cosine functions, students sketch graphs showing key features, which include intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maxima and minima; symmetries; end behavior; and periodicity.

Lesson Notes

Students spend this lesson exploring the graphs of the sine and cosine functions. The lesson opens with an Exploratory Challenge where students create the graphs of the sine and cosine functions using spaghetti. The purpose of this activity is to allow students to discover the key features of the sine and cosine functions and to connect the graphs to the measurements on the unit circle. If time is short, Exploratory Challenge 1 can be omitted. Throughout the lesson, an emphasis should be placed on the shape of these curves that they have seen before when creating the graphs of the height and co-height functions in Lessons 1 and 2. For example, the arcs are not semicircular or parabolic; $\sin(45°)$ is not halfway between $\sin(0°)$ and $\sin(90°)$, etc. Students should spend time comparing and contrasting the key features of the two graphs. In this lesson, we are still working with measuring rotations in degrees. Emphasize to students that the graphs they are creating are not being drawn to scale—the horizontal axis and vertical axis have wildly different scales so that students can get a good sense of the shape of the sine and cosine graphs. This is critical for the development of radian measure in the next lesson. Students should still use θ to represent the degrees of rotation that locate the point (x_θ, y_θ) on the unit circle that define the values of sine and cosine; this is primarily to avoid using the variable x to represent both the coordinate x_θ on the unit circle and the independent variable of the trigonometric functions.

Classwork

Exploratory Challenge 1 (18 minutes)

Students work in groups to create either the graph of the sine function or the cosine function using pieces of uncooked spaghetti. The spaghetti can be replaced by narrow strips of construction paper if necessary. Each group will need the following supplies:

- Blank unit circle
- Yarn
- Marker
- Spaghetti (uncooked, roughly 30 strands per group, allowing extra for breakage)
- Piece of paper or poster board that measures at least $9" \times 30"$
- Glue stick

Scaffolding:

If students are struggling, provide some additional support.

- Model the activity for the class or for a small group.
- Read the directions aloud to the class and ask students to summarize.

Assign groups either the sine or cosine function by circling one of the two on the instruction sheet before distributing to groups. Encourage groups to divide the work in order to complete the activity.

As students work in groups, circulate to monitor their progress. As groups get to the part where they measure the sine or cosine, make sure they understand what the pieces of spaghetti represent. Remind them that for a point on the unit circle that has been rotated through θ degrees, $\cos(\theta°)$ is the x-coordinate of the point, and $\sin(\theta°)$ is the y-coordinate of the point. Watch that students are remembering to measure distance along the perpendicular and to place negative values below the x-axis. Then, allow students to present the graphs to the class. Graphs can be displayed on the walls of your classroom.

Exploratory Challenge 1

Your group will be graphing: $f(\theta) = \sin(\theta°)$ $g(\theta) = \cos(\theta°)$

The circle on the handout is a unit circle, meaning that the length of the radius is one unit.

1. **Mark axes on the poster board, with a horizontal axis in the middle of the board and a vertical axis near the left edge, as shown.**

2. **Measure the radius of the circle using a ruler. Use the length of the radius to mark 1 and -1 on the vertical axis.**
3. **Wrap the yarn around the circumference of the circle starting at 0. Mark each $15°$ increment on the yarn with the marker. Unwind the yarn and lay it on the horizontal axis. Transfer the marks on the yarn to corresponding increments on the horizontal axis. Label these marks as $0, 15, 30, \ldots, 360$.**
4. **Record the number of degrees of rotation θ on the horizontal axis of the graph, and record the value of either $\sin(\theta°)$ or $\cos(\theta°)$ on the vertical axis. Notice that the scale is wildly different on the vertical and horizontal axes.**
5. **If you are graphing $g(\theta) = \cos(\theta°)$: For each θ marked on your horizontal axis, beginning at 0, use the spaghetti to measure the horizontal displacement from the vertical axis to the relevant point on the unit circle. The horizontal displacement is the value of the cosine function. Break the spaghetti to mark the correct length, and place it vertically at the appropriate tick mark on the horizontal axis.**
6. **If you are graphing $f(\theta) = \sin(\theta°)$: For each θ marked on your horizontal axis, beginning at 0, use the spaghetti to measure the vertical displacement from the horizontal to the relevant point on the unit circle. The vertical displacement is the value of the sine function. Break the spaghetti to mark the correct length, and place it vertically at the appropriate tick mark on the horizontal axis.**
7. **Remember to place the spaghetti below the horizontal axis when the value of the sine function or the cosine function is negative. Glue each piece of spaghetti in place.**
8. **Draw a smooth curve that connects the points at the end of each piece of spaghetti.**

Discussion (4 minutes)

After the groups have presented their graphs, have the following discussion. Much of this will be formalized in Exploratory Challenge 2. The purpose here is to informally examine the graphs. Use GeoGebra, Wolfram Alpha or desmos.com (a free online graphing calculator) to display the graphs and compare to the ones made by students.

- How are the graphs of the sine and cosine functions alike?
  - *They have the same basic shape. They have the same maximum and minimum values.*

- Could I get the graph of the sine function by shifting the graph of the cosine function?
  - *It appears that I could get the graph of the sine function by shifting the graph of the cosine function to the right by* 90.
- If we extended the horizontal axis, what would the graph of the sine function look like between 360 and 720? What about from 720 to 1080?
  - *The graph would look the same as it did from* 0 *to* 360 *because the values of sine (or cosine) will just repeat.*
- What if we extended the graph in the negative direction?
  - *Again, the graph would continue to repeat the same pattern on every interval of length* 360.

MP.7

Exploratory Challenge 2 (15 minutes)

Students work in groups to graph $f(\theta) = \sin(\theta°)$ and $g(\theta) = \cos(\theta°)$ by making a table of values and then analyzing the two graphs. If time is short, assign the groups who graphed the sine function in the last exercise to graph the cosine function now and vice versa. Ensure students have access to technology to assist with completing the table.

Exploratory Challenge 2

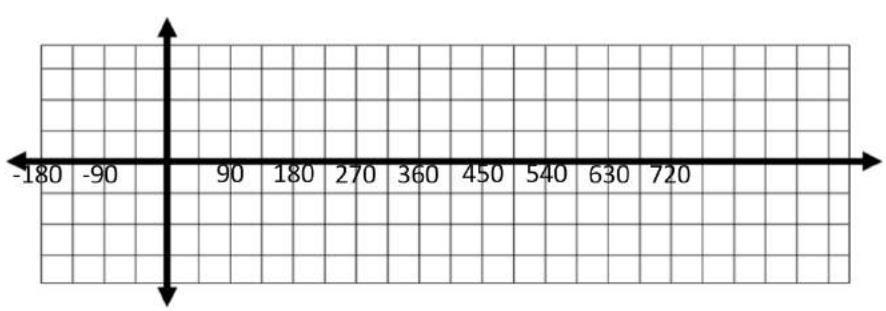
Part I: Consider the function $f(\theta) = \sin(\theta°)$.

a. **Complete the following table by using the special values learned in Lesson 4. Give values as approximations to one decimal place.**

θ, in degrees	0	30	45	60	90	120	135	150	180
$\sin(\theta°)$	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0

θ, in degrees	210	225	240	270	300	315	330	360
$\sin(\theta°)$	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0

b. **Using the values in the table, sketch the graph of the sine function on the interval $[0, 360]$.**



c. **Extend the graph of the sine function above so that it is graphed on the interval from $[-180, 720]$.**

d. **For the interval $[-180, 720]$, describe the values of θ at which the sine function has relative maxima and minima.**

Relative maxima occur at 90 *and* 450. *Relative minima occur at* -90, 270, *and* 630.

Scaffolding:

If students are struggling, start with a series of easier questions once they have completed the graph.

- What do you notice about the graph?
- What features does it have?
- How would you describe it to someone who can't see it?

As an extension, ask

- How would the graph of cosine be different?

e. **For the interval $[-180, 720]$, describe the values of θ for which the sine function is increasing and decreasing.**

Increasing: $-90 < \theta < 90$, $270 < \theta < 450$, and $630 < \theta < 720$

Decreasing: $-180 < \theta < -90$, $90 < \theta < 270$, and $450 < \theta < 630$

f. **For the interval $[-180, 720]$, list the values of θ at which the graph of the sine function crosses the horizontal axis.**

$-180, 0, 180, 360, 540, 720$

g. **Describe the end behavior of the sine function.**

The sine function will continue to oscillate between -1 and 1 as $\theta \to \infty$ or $\theta \to -\infty$.

h. **Based on the graph, is sine an odd function, even function, or neither? How do you know?**

Sine is an odd function because the graph is symmetric with respect to the origin.

i. **Describe how the sine function repeats.**

The sine function repeats the same pattern on every interval of length 360 because $360°$ is one full turn. After that, the values of sine just continue to repeat.

Debrief to make sure students have correctly graphed the sine function and identified its key features.

- At which values of θ is the sine function at a maximum?
  - $90, 450, -270$ *(Answers may vary.)*
- Are the values you listed the only places where the sine function is at a maximum?
  - *No, the graph will continue to repeat. Every time we get to a rotation value that produces a point that is at the top of the circle, the graph will reach a maximum point.*
- How could we write the answer in general terms to include all values of θ at which the sine function is at a maximum? (Lead students to the equation $\theta = 90 + 360n$ for all integers n.)
  - *The sine function is at a maximum at $90, 90 + 360, 90 + 720, \ldots$ and $90 - 360, 90 - 720, \ldots$, which is 90 plus some multiple of 360.*

Ask groups to now do the same for the values of θ for which the sine function is at a minimum and where the graph of the sine function crosses the horizontal axis. Then, share responses.

- At which values of θ is the sine function at a minimum?
  - *At $\theta = 270 + 360n$ for all integers n (or, equivalently, $\theta = -90 + 360n$)*
- At which values of θ does the graph of the sine function cross the horizontal axis?
  - *At $\theta = 360n$ for all integers n*

MP.2

- How could we describe the end behavior of the sine function?
  - *As $\theta \to \infty$, the sine function does not approach a specific value or go up toward ∞ or down toward $-\infty$. It just keeps repeating the same pattern over and over. The same thing happens as $\theta \to -\infty$.*
- How far do you have to go on the graph before the pattern repeats? Why?
  - *The pattern repeats after 360 because $360°$ is one full turn. After that we are just repeating the same values around the circle.*

Before students begin to work on Part II of this challenge, ask them to make a conjecture about how the graph of the cosine function will be the same as the sine function and how it will be different. Discuss as a class or ask students to share with a partner. As they begin to work on Part II, instruct them to list the answers to (d) and (f) that are apparent from the graph and then to write the answer in general terms.

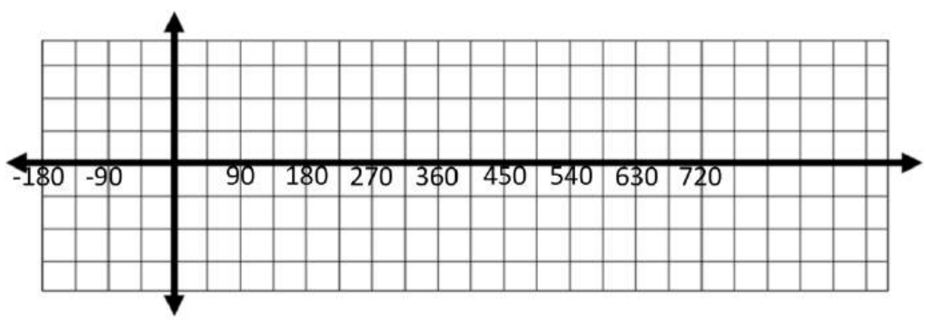
Part II: Consider the function $g(\theta) = \cos(\theta°)$.

a. Complete the following table giving answers as approximations to one decimal place.

θ, in degrees	0	30	45	60	90	120	135	150	180
$\cos(\theta°)$	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1

θ, in degrees	210	225	240	270	300	315	330	360
$\cos(\theta°)$	-0.9	-0.7	-0.5	0	0.5	0.7	0.9	1

b. Using the values in the table, sketch the graph of the cosine function on the interval $[0, 360]$.



c. Extend the graph of the cosine function above so that it is graphed on the interval from $[-180, 720]$.

d. For the interval $[-180, 720]$, describe the values of θ at which the cosine function has relative maxima and minima.

Maxima occur at $0, 360, 720$. *Minima occur at $-180, 180, 540$.*

e. For the interval $[-180, 720]$, describe the values of θ for which the cosine function is increasing and decreasing.

Increasing: $-180 < \theta < 0$, $180 < \theta < 360$, and $540 < \theta < 720$

Decreasing: $0 < \theta < 180$ and $360 < \theta < 540$

f. **For the interval $[-180, 720]$, list the values of θ at which the graph of the cosine function crosses the horizontal axis.**

$-90, 90, 270, 450, 630$

g. **Describe the end behavior of the graph of the cosine function.**

The cosine function will continue to oscillate between -1 and 1 as $\theta \to \infty$ or $\theta \to -\infty$.

h. **Based on the graph, is cosine an odd function, even function, or neither? How do you know?**

Cosine is an even function because the graph is symmetric with respect to the vertical axis.

i. **Describe how the cosine function repeats.**

The cosine function repeats the same pattern on intervals of length 360 because $360°$ is one full turn. After that, the values of cosine just continue to repeat.

j. **How are the sine function and cosine function related to each other?**

The two functions are horizontal translations of each other. We could shift the graph of either function horizontally to line up with the graph of the other function.

- At which values of θ is the cosine function at a maximum?
  - At $\theta = 360n$ *for all integers* n
- At which values of θ is the cosine function at a minimum?
  - At $\theta = 180 + 360n$ *for all integers* n
- Where does the graph of the cosine function cross the horizontal axis?
  - At $\theta = 90 + 360n$ *and* $\theta = 270 + 360n$ *for all integers* n *(This can also be expressed as* $\theta = 90 + 180n$*, for all integers* n*.)*
- What are some ways in which the graphs of the sine and cosine functions are alike?
  - *They have the same type of end behavior, they both repeat the same pattern on an interval length of* 360*, they have the same basic shape, and they are horizontal shifts of each other.*
- If we wanted to do a quick graph of the sine or cosine function showing all the key points, what values of θ would be the most important to use? Why?
  - *It would be important to use the values of* θ *that correspond to points located on the* x*-axis and* y*-axis of the unit circle. They are the relative maxima and minima of sine and cosine and also where the graphs of sine and cosine will cross the horizontal axis.*

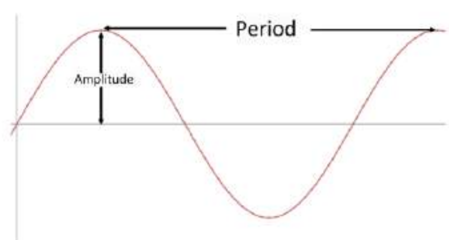
Closing (4 minutes)

Have students read the definitions in the lesson summary. Ask them to identify the period and amplitude of the sine and cosine functions and then share responses with a partner. Mastery of these terms is not expected at this point. The intent is to begin to use the vocabulary associated with the sine and cosine functions.

- Why are sine and cosine examples of periodic functions?
- What is the period of each function? Why?
- How is amplitude measured? What is the amplitude of each function?

Lesson Summary

- **A function f whose domain is a subset of the real numbers is said to be *periodic with period* $P > 0$ if the domain of f contains $x + P$ whenever it contains x, and if $f(x + P) = f(x)$ for all real numbers x in its domain.**
- **If a least positive number P exists that satisfies this equation, it is called the *fundamental period* or, if the context is clear, just the *period* of the function.**
- **The *amplitude* of the sine or cosine function is half of the distance between a maximal point and a minimal point of the graph of the function.**



Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 8: Graphing the Sine and Cosine Functions

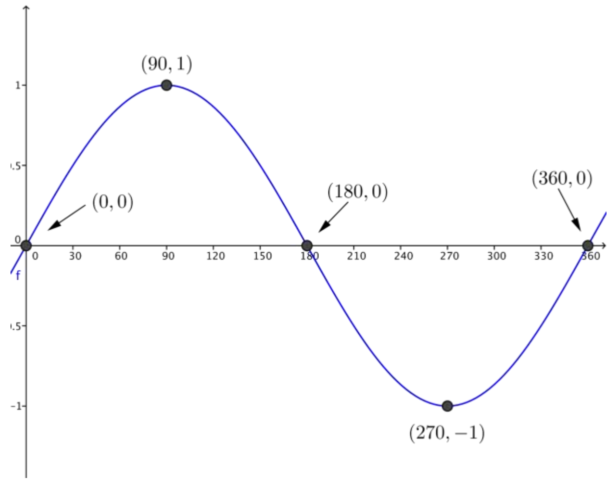
Exit Ticket

1. Sketch a graph of the sine function on the interval $[0, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Mark the coordinates of the maximum and minimum points and the intercepts.

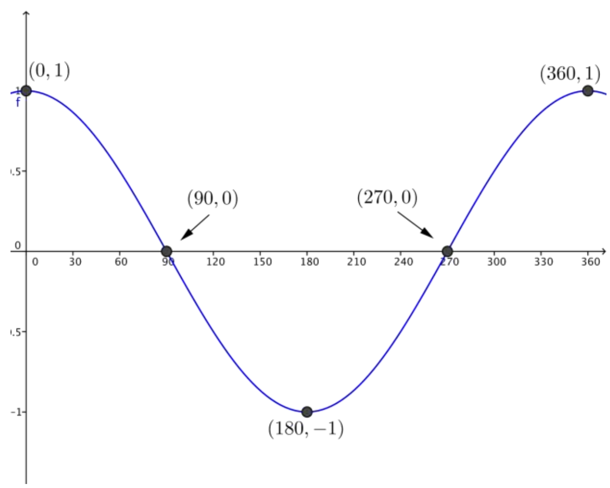
2. Sketch a graph of the cosine function on the interval $[0, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Mark the coordinates of the maximum and minimum points and the intercepts.

Exit Ticket Sample Solutions

1. **Sketch a graph of the sine function on the interval $[0, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Mark the coordinates of the maximum and minimum points and the intercepts.**



2. **Sketch a graph of the cosine function on the interval $[0, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Mark the coordinates of the maximum and minimum points and the intercepts.**



Problem Set Sample Solutions

1. Graph the sine function on the interval $[-360, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Then, use the graph to answer each of the following questions.

   a. On the interval $[-360, 360]$, what are the relative minima of the sine function? Why?

   The sine function has relative minima at -90 and 270 because when rotated by $-90°$ or $270°$, the initial ray intersects the unit circle at the bottom of the circle; therefore, the sine is at its smallest possible value.

   b. On the interval $[-360, 360]$, what are the relative maxima of the sine function? Why?

   The sine function has relative maxima at -270 and 90 because when rotated by $-270°$ or $90°$, the initial ray intersects the unit circle at the top of the circle; therefore, the sine is at its largest possible value.

   c. On the interval $[-360, 360]$, for what values of θ is $\sin(\theta°) = 0$? Why?

   The sine function has a value of 0 at -360, -180, 0, 180, and 360 because when rotated by any of these degree measurements, the initial ray intersects the unit circle at a point on the x-axis; therefore, $\sin(\theta°)$, which measures the height of the point on the circle, is equal to 0.

   d. If we continued to extend the graph in either direction, what would it look like? Why?

   It would repeat the same pattern on intervals of length 360 in either direction. So, from 360 to 720, the graph would look exactly like it does from 0 to 360, as it also would from 720 to 1080, and so on. Also, from -360 to 0, the graph would look the same as it does from 0 to 360, as it also would from -720 to -360, and so on.

   e. Arrange the following values in order from smallest to largest by using their location on the graph.

   $\sin(170°)$ $\sin(85°)$ $\sin(-85°)$ $\sin(200°)$

   $\sin(-85°)$ $\sin(200°)$ $\sin(170°)$ $\sin(85°)$

   f. On the interval $(90, 270)$, is the graph of the sine function increasing or decreasing? Based on that, name another interval not included in $(90, 270)$ where the sine function must have the same behavior.

   Decreasing; $(450, 630)$

2. Graph the cosine function on the interval $[-360, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Then, use the graph to answer each of the following questions.

   a. On the interval $[-360, 360]$, what are the relative minima of the cosine function? Why?

   For the cosine function, a relative minimum of -1 occurs at -180 and 180 because when rotated by $-180°$ or $180°$, the initial ray intersects the unit circle at the leftmost point of the circle; therefore, the cosine is at its smallest possible value.

   b. On the interval $[-360, 360]$, what are the relative maxima of the cosine function? Why?

   For the cosine function, a relative maximum of 1 occurs at -360, 0, and 360 because when rotated by $-360°$, $0°$, or $360°$, the initial ray intersects the unit circle at the rightmost point of the circle; therefore, the cosine is at its largest possible value.

c. On the interval $[-360, 360]$, for what values of θ is $\cos(\theta°) = 0$? Why?

The cosine function has a value of 0 at -270, -90, 90, and 270 because when rotated by any of these degree measurements, the initial ray intersects the unit circle at a point on the y-axis; therefore, $\cos(\theta°)$, which measures the horizontal position of the point on the circle, is equal to 0.

d. If we continued to extend the graph in either direction, what would it look like? Why?

It would repeat the same pattern on intervals of length 360 in either direction. So from 360 to 720, the graph would look exactly like it does from 0 to 360, as it also would from 720 to 1080, and so on. Also, from -360 to 0, the graph would look the same as it does from 0 to 360, as it also would from -720 to -360, and so on.

e. What can be said about the end behavior of the cosine function?

Like the sine function, the cosine function continues to oscillate between the values of -1 and 1 as θ approaches ∞ or $-\infty$.

f. Arrange the following values in order from smallest to largest by using their location on the graph.

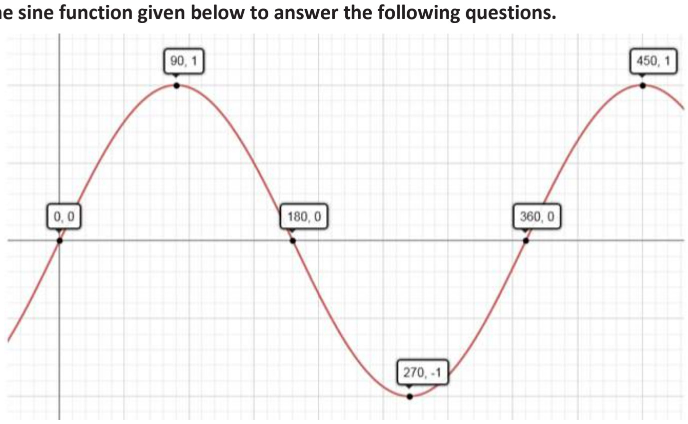
$\cos(135°)$ $\cos(85°)$ $\cos(-15°)$ $\cos(190°)$

$\cos(190°)$ $\cos(135°)$ $\cos(85°)$ $\cos(-15°)$

3. Write a paragraph comparing and contrasting the sine and cosine functions using their graphs and end behavior.

The graphs of sine and cosine have the same domain, range, amplitude, and period. Neither graph has end behavior because both oscillate between the values of -1 and 1 as θ approaches ∞ or $-\infty$. The graphs have different y-intercepts (starting points) because at $0°$ sine is 0 and cosine is 1. When sine is at its maximum or minimum, the cosine function is 0 and has a horizontal intercept. The graphs are horizontal translations of each other, so the graph of cosine can be made to look like sine by using a horizontal shift and vice versa.

4. Use the graph of the sine function given below to answer the following questions.



a. Desmond is trying to determine the value of $\sin(45°)$. He decides that since 45 is halfway between 0 and 90, $\sin(45°) = \frac{1}{2}$. Use the graph to show him that he is incorrect.

Using the graph, the y-value at $\theta = 45$ is above 0.5, so $\sin(45°) > \frac{1}{2}$. That is, the graph is not linear between the point $(0, 0)$ and the point $(90, 1)$.

b. Using the graph, complete each statement by filling in the symbol $>$, $<$, or $=$.

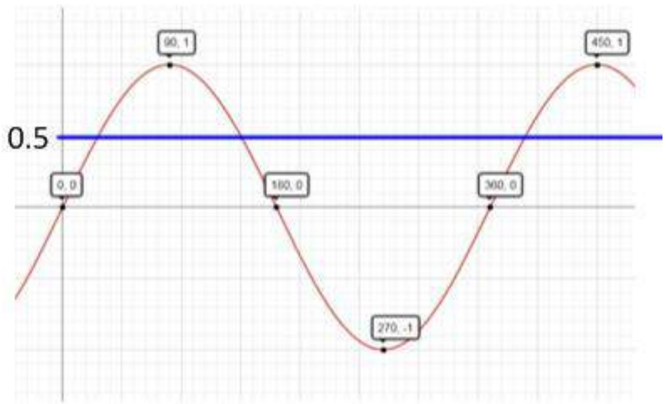
i. $\sin(250°)$ $=$ $\sin(290°)$

ii. $\sin(25°)$ $<$ $\sin(85°)$

iii. $\sin(140°)$ $>$ $\sin(160°)$

c. On the interval $[0, 450]$, list the values of θ such that $\sin(\theta°) = \frac{1}{2}$.

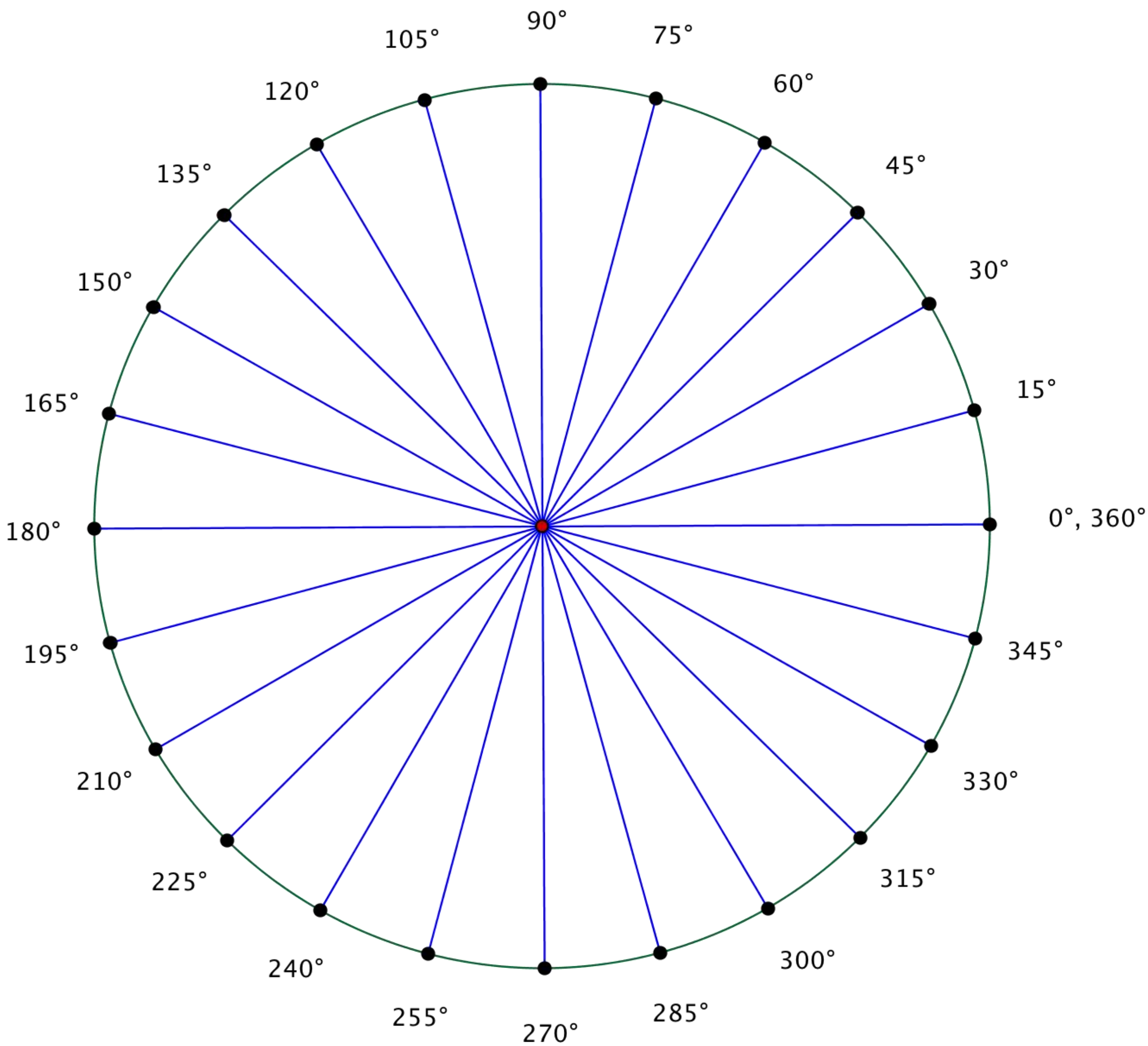
$30, 150, 390$



d. Explain why there are no values of θ such that $\sin(\theta°) = 2$.

The sine function ranges in value from -1 to 1. Since the maximum value of the sine function is 1 (when the terminal ray of the rotation intersects the top of the circle), it is impossible for there to be a value θ such that $\sin(\theta°) = 2$.

Exploratory Challenge Unit Circle Diagram



Lesson 9: Awkward! Who Chose the Number 360, Anyway?

Student Outcomes

- Students explore horizontal scalings of the graph of $y = \sin(x^\circ)$.
- Students convert between degrees and radians.

Lesson Notes

In this lesson, justification is given for changing from using degree measure for rotation to radian measure. The main argument for this change is that the graph of the sine and cosine functions are ridiculously flat if graphed on a square grid, so there is the need to change the horizontal scale. Graphing calculators are used to investigate different rescalings of the sine graph, and students see that the graph of $f(x) = \sin\left(\frac{180}{\pi} x^\circ\right)$ aligns with the diagonal line $y = x$ near the origin.

This reason given may seem somewhat artificial, and, from the students' perspective, it is. It is true that the reason radians are used instead of degrees is that when using radians, the function $g(x) = \frac{\sin(x)}{x}$ has slope 1 near the origin, but this is not the entire story. The fact is that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ leads to the derivative formula $\frac{d}{dx}\sin(x) = \cos(x)$, which greatly simplifies derivative calculations with the sine and cosine functions in calculus and beyond. Deciding whether to discuss these reasons with students should be based upon their readiness for these advanced ideas.

In any case, in Exercises 1–4 in this lesson, students graph various functions $f(x) = \sin(kx^\circ)$, looking for a function whose graph is diagonal near the origin. This provides students with an opportunity to employ MP.8 as they make generalizations about k by repeatedly graphing these functions. This exploration gives students a head start on the work in Lesson 11, in which students explore the effects of the parameters A, w, h, and k on the graph of general sinusoidal functions of the form $f(x) = A\sin\big(w(x-h)\big) + k$.

In this lesson, students will finally be equipped to give definitions of the sine and cosine function in terms of radians.

> **SINE FUNCTION (description):** The *sine function*, $\sin: \mathbb{R} \to \mathbb{R}$, can be defined as follows: Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\sin(\theta)$ is y_θ.
>
> **COSINE FUNCTION (description):** The *cosine function*, $\cos: \mathbb{R} \to \mathbb{R}$, can be defined as follows: Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\cos(\theta)$ is x_θ.

In the definitions of the trigonometric functions, θ is always a real number. That is, the input for the sine function is a number and not a quantity such as 30 radians. How the real number θ is used is actually part of the definition: The rule for finding $\sin(\theta)$ states to rotate the initial ray by θ radians about the origin. The number θ is given a quantitative meaning (radians) by the definition.

Because there are two measurement systems for rotational measure, degrees and radians, there are, therefore, two different definitions of each trigonometric function—one for degrees and one for radians. The rigorous thing to do in this situation is to notate each definition differently, say by $\sin_{\text{deg}}(40)$ to refer to a rotation by 40 degrees and $\sin_{\text{rad}}(40)$ to refer to a rotation by 40 radians. Using two notations, of course, is unforgivably pedantic, and mathematicians long ago decided to notate the difference between the definitions of these two functions in a subtle way.

Here's the rule of thumb: $\sin(\theta)$ always refers to the value of the sine function after rotating the initial ray by θ radians. Once radians are defined in Algebra II, this becomes the standard way to refer to almost all rotational measures and sine functions from that point on (including in precalculus and university-level calculus). To refer to the degree definition of the sine function, indicate the degree symbol on the number, as in $\sin(45°)$. Note, however, that the degree symbol in this notation refers to which definition of the sine function is being used, not that the sine function is accepting a quantity of $45°$ as input. In this notation, the sine function is still thought of as accepting a number, say 45, but 45 is being put into the blank in the notation $\sin(\_\_°)$. The degree symbol is referring to the definition and isn't really part of the input. Subtle? Yes. But it is also so natural that students will never really question its use. There is no need to explicitly explain this subtlety; just be clear about how to use and apply the notation correctly and consistently.

Materials

For the Problem Set, students need access to a radian protractor. An inexpensive way to obtain these is to copy the images from the last page onto transparencies, cut out the protractors, and distribute to students.

Classwork

Opening Exercise (8 minutes)

The first step in understanding why there is a need to change how angles of rotation are measured is to try to graph the sine function on a grid with the same horizontal and vertical scale. Allow students to struggle with this task and to come to the conclusion that either the horizontal or the vertical scale needs to change in order to be able to even see the graph of $y = \sin(x°)$.

Scaffolding:

Students above grade level can plot points in 15 degree increments to get a better image of the graph.

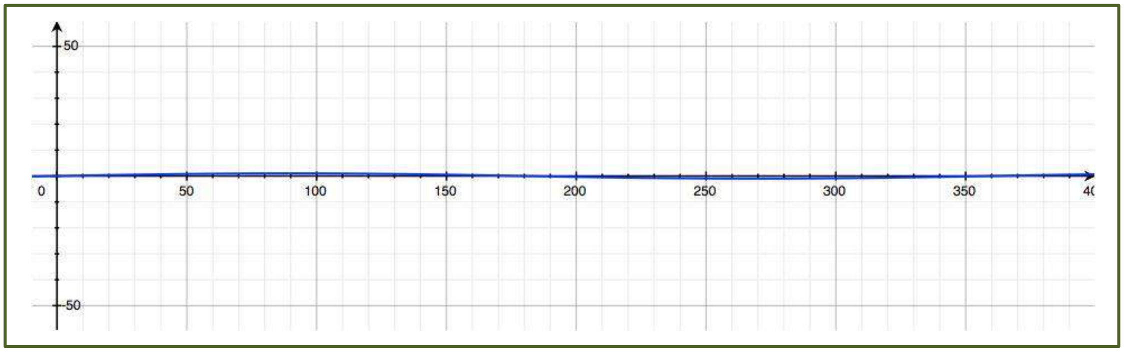
Opening Exercise

Let's construct the graph of the function $y = \sin(x°)$, where x is the measure of degrees of rotation. In Lesson 5, we decided that the domain of the sine function is all real numbers and the range is $[-1, 1]$. Use your calculator to complete the table below with values rounded to one decimal place, and then graph the function on the axes below. Be sure that your calculator is in degree mode.

x, in degrees	$y = \sin(x°)$
0	0.0
30	0.5
45	0.7
60	0.9
90	1.0
120	0.9

x, in degrees	$y = \sin(x°)$
135	0.7
150	0.5
180	0.0
210	-0.5
225	-0.7
240	-0.9

x, in degrees	$y = \sin(x°)$
270	-1.0
300	-0.9
315	-0.7
330	-0.5
360	0.0



Discussion (4 minutes)

This discussion should lead to the conclusion that the best way to “fix” the problem of graphing $y = \sin(x°)$ while retaining a one-to-one ratio of the scales on the axes would be to perform a horizontal scaling to compress the graph along the horizontal axis. The first step is to guide students to recognize the nature of the problem they’re facing.

- What do you notice about the graph that you created in the Opening Exercise?
  - *Creating a useful graph is impossible on the set of axes provided.*
- In Lesson 30 of Module 1, we performed transformations on a parabola that changed its horizontal scale. How did we do that?
  - *When we graphed $y = x^2$ and $y = \left(\frac{1}{k}x\right)^2$, the second graph was stretched horizontally by a factor of k.*
- What horizontal scaling can we perform to compress the parabola $y = x^2$ to make the graph twice as narrow as the original graph?
  - $y = (2x)^2$
- What might be a reasonable transformation to perform on the sine function to make the graph narrower?
  - *Any transformation of the form $y = \sin(kx°)$ for $k \geq 1$ will work; students may suggest $y = \sin(2x°)$ or $y = \sin(10x°)$.*
- Do we really know which transformation would be the best? Maybe we should stop to think about what we want to transform this graph into before we proceed.

Exercises 1–4 (8 minutes)

Place students in small groups and keep them working in these groups for all of the exercises in this lesson.

MP.8

Exercises 1–5

Set your calculator's viewing window to $0 \leq x \leq 10$ and $-2.4 \leq y \leq 2.4$, and be sure that your calculator is in degree mode. Plot the following functions in the same window:

$y = \sin(x°)$

$y = \sin(2x°)$

$y = \sin(10x°)$

$y = \sin(50x°)$

$y = \sin(100x°)$

1. **This viewing window was chosen because it has close to the same scale in the horizontal and vertical directions. In this viewing window, which of the five transformed sine functions most clearly shows the behavior of the sine function?**

   Students may answer $y = \sin(50x°)$ or $y = \sin(100x°)$. (Either answer is reasonable.)

2. **Describe the relationship between the steepness of the graph $y = \sin(kx°)$ near the origin and the value of k.**

   As we increase k, the steepness of the graph $y = \sin(kx°)$ near the origin increases.

3. **Since we can control the steepness of the graph $y = \sin(kx°)$ near the origin by changing the value of k, how steep might we want this graph to be? What is your *favorite* positive slope for a line through the origin?**

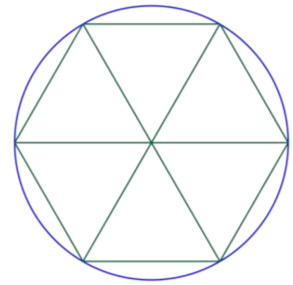
   It would make sense to try to get the steepness at the origin to be the same as the diagonal line $y = x$, which has slope 1.

4. **In the same viewing window on your calculator, plot $y = x$ and $y = \sin(kx°)$ for some value of k. Experiment with your calculator to find a value of k so that the steepness of $y = \sin(kx°)$ matches the slope of the line $y = x$ near the origin. You may need to change your viewing window to $0 \leq x \leq 2$ and $0 \leq y \leq 1$ to determine the best value of k.**

   The graph of $y = \sin(57x°)$ has nearly the same steepness as the diagonal line with equation $y = x$.

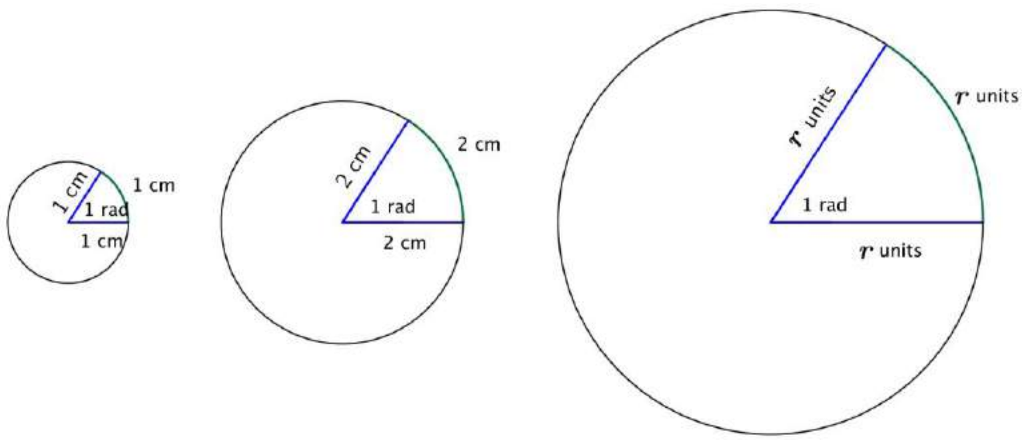
Discussion (4 minutes)

- Which values of k produce graphs of $y = \sin(kx°)$ that are close to the graph of $y = x$ near the origin?
  - *Responses will vary; they should be near $k = 57$.*
- It looks like choosing $k = 57$ is close to what we want. But why 57? Something is strange here! And, indeed, there is something both surprising and natural about what the value of k truly is.
- First, let's review our basic system of measurement for rotation. We could take the entire circle as the unit of rotational measure; this is known as a *turn*. Then, a rotation can be expressed as a fraction of a turn. For example, $\frac{1}{4}$ turn would correspond to a right angle, and $\frac{1}{2}$ of a turn would correspond to a straight angle.
- Instead, we more commonly use a small unit called a *degree*. We divide the circle into 360 arcs of equal length, and then the central angle subtended by one of these arcs has measure 1 degree. Then, a turn measures 360°.

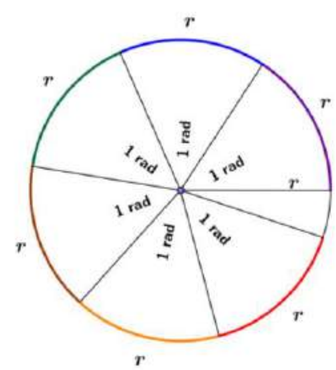


- Who came up with our current system of using $360°$ in a turn? Remember the ancient Babylonians who made all of those astronomical observations that led to the discovery of trigonometry? They are also the ones responsible for our system of measuring rotations and angles. It appears that the Babylonians subdivided the circle using the angle of an equilateral triangle as the basic unit. Since they used a base-60 number system, they divided each angle of the equilateral triangle into 60 smaller units, each with measure 1 degree, giving 360 degrees in a turn. Each degree is subdivided into 60 minutes, and each minute is subdivided into 60 seconds.
- For our purposes now, using $360°$ in a turn is cumbersome. Instead of basing our measurement system on an arbitrary number like 360, we will instead use a system in which the measures of angles and rotations are determined by the length of the corresponding arc of a unit circle.

- **A circle is defined by a point and a radius. If we start with a circle of any radius and look at a sector of that circle with an arc length equal to the length of the radius, then the central angle of that sector is always the same size. We define a *radian* to be the measure of that central angle and denote it by 1 rad.**



- **Thus, a radian measures how far one radius will wrap around the circle. For any circle, it takes $2\pi \approx 6.3$ radius lengths to wrap around the circumference. In the figure, 6 radius lengths are shown around the circle, with roughly 0.3 radius lengths left over.**



Exercise 5 (2 minutes)

Allow students time to discuss this with a partner in order to make the connection between the $57°$ measured in this exercise and the $k = 57$ scale factor that was discovered in Exercise 4.

> **5. Use a protractor that measures angles in degrees to find an approximate degree measure for an angle with measure 1 rad. Use one of the figures from the previous discussion.**
>
> *We find that the degree measure of an angle that has measure 1 rad is approximately $57°$.*

Examples 1–4 (4 minutes)

Instead of emphasizing conversion formulas for switching between degrees and radians, emphasize that both systems can be thought of as fractions of a turn. Thus, a $60°$ rotation is $\frac{1}{6}$ of a turn, which is $\frac{\pi}{3}$ radians, and π radians is a half-turn, which is $180°$.

Examples 1–4

1. **Convert from degrees to radians: $45°$**

$$\begin{aligned} 45° &= \frac{1}{8} \text{ turn} \\ &= \frac{1}{8}(2\pi) \text{ rad} \\ &= \frac{\pi}{4} \text{ rad} \end{aligned}$$

2. **Convert from degrees to radians: $33°$**

$$\begin{aligned} 33° &= \frac{33}{360} \text{turn} \\ &= \left(\frac{33}{360}\right) 2\pi \text{ rad} \\ &= \left(\frac{11}{60}\right) \pi \text{ rad} \end{aligned}$$

3. **Convert from radians to degrees: $-\frac{\pi}{3}$ rad**

$$\begin{aligned} -\frac{\pi}{3} \text{ rad} &= -\left(\frac{1}{6}\right) \text{turn} \\ &= -\left(\frac{1}{6}\right) 360° \\ &= -60° \end{aligned}$$

4. **Convert from radians to degrees: $\frac{19\pi}{17}$ rad**

$$\begin{aligned} \frac{19\pi}{17} \text{ rad} &= \left(\frac{19}{17}\right)\left(\frac{1}{2}\right) \text{turn} \\ &= \left(\frac{19}{34}\right) 360° \\ &\approx 201° \end{aligned}$$

Exercise 6 (3 minutes)

Have students perform the following conversions either alone or in small groups to complete this chart. Circulate around the room to monitor student progress, especially for the last conversion that is critical to the conclusion students should make in this lesson.

Exercises 6–7

6. **Complete the table below, converting from degrees to radians or from radians to degrees as necessary. Leave your answers in exact form, involving π.**

Degrees	Radians
$45°$	$\frac{\pi}{4}$
$120°$	$\frac{2\pi}{3}$
$-150°$	$-\frac{5\pi}{6}$
$270°$	$\frac{3\pi}{2}$
$450°$	$\frac{5\pi}{2}$
$x°$	$\left(\frac{\pi}{180}\right)x$
$\left(\frac{180}{\pi}\right)x°$	x

Exercise 7 (3 minutes)

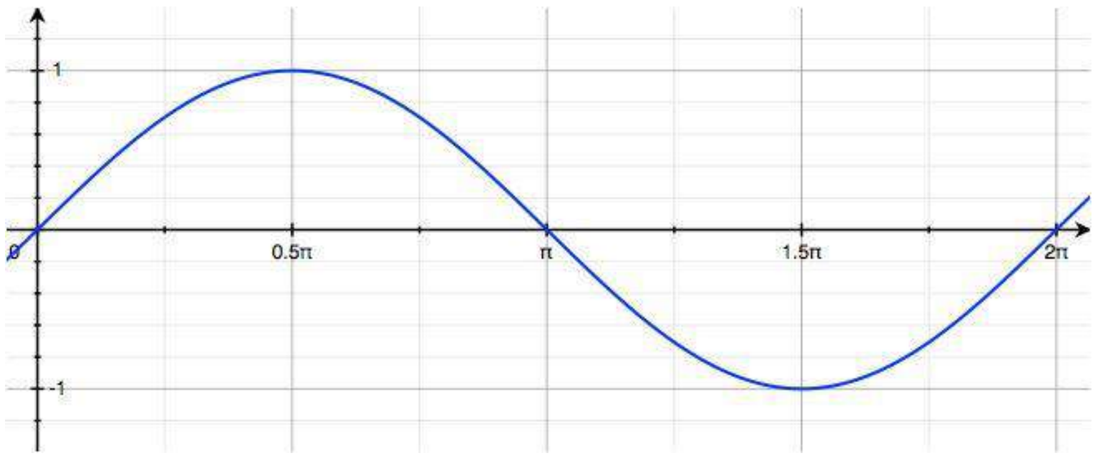
This question ties together the previous exploration looking for a transformed sine function that is diagonal near the origin and our newly defined radian measure for angles. Allow students to continue to work in their groups on these questions.

7. **On your calculator, graph the functions $y = x$ and $y = \sin\left(\frac{180}{\pi}x°\right)$. What do you notice near the origin? What is the decimal approximation to the constant $\frac{180}{\pi}$ to one decimal place? Explain how this relates to what we've done in Exercise 4.**

The graph of $y = \sin\left(\frac{180}{\pi}x°\right)$ is nearly identical to the graph of $y = x$ near the origin. On the calculator, we see that $\frac{180}{\pi} \approx 57.3$ so that $\sin\left(\frac{180}{\pi}x°\right) \approx \sin(57x°)$. This is the function we were looking for in Exercise 4.

Discussion (2 minutes)

If degrees are changed to radians, then the expression $\sin\left(\frac{180}{\pi}x^\circ\right)$, where x is measured in degrees, becomes $\sin(x)$, where x is measured in radians. Then one period of the graph of $y = \sin(x)$ on a grid with the same scale on the horizontal and vertical axes now looks like this:



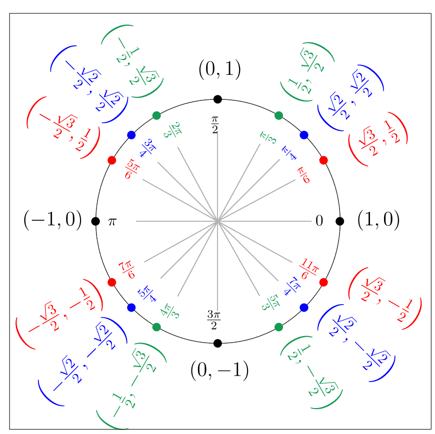
From this point forward, trigonometric functions will always be graphed using radians for measuring rotation instead of degrees. Besides the discovery that the graph of the sine function is much easier to create and use in radians, it turns out that radians make many calculations much easier in later work in mathematics.

Closing (3 minutes)

Ask students to summarize the main points of the lesson either in writing, to a partner, or as a class.

- A *radian* is the measure of the central angle of a sector of a circle with arc length of one radius length.
- If there is no degree symbol or specification, then the use of radians is implied.
- There are 2π radians in a $360°$ rotation, also known as a *turn*, so degrees are converted to radians and radians to degrees by:

  $2\pi \text{ rad} = 1 \text{ turn} = 360°$.
- From this point forward, all work will be done with radian measures exclusively for rotation and as the independent variables in the trigonometric functions. The diagram is nearly the same as the one for the sine and cosine functions in Lesson 4, but this time it is labeled with rotations measured in radians.



- Use the diagram to find $\cos\left(\frac{7\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$
  - $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
  - $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Lesson Summary

- **A *radian* is the measure of the central angle of a sector of a circle with arc length of one radius length.**
- **There are 2π radians in a $360°$ rotation, also known as a *turn*, so degrees are converted to radians and radians to degrees by:**

$$2\pi \text{ rad} = 1 \text{ turn} = 360°.$$

- **SINE FUNCTION (description): The *sine function*, $\sin: \mathbb{R} \to \mathbb{R}$, can be defined as follows: Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\sin(\theta)$ is y_θ.**
- **COSINE FUNCTION (description): The *cosine function*, $\cos: \mathbb{R} \to \mathbb{R}$, can be defined as follows: Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\cos(\theta)$ is x_θ.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 9: Awkward! Who Chose the Number 360, Anyway?

Exit Ticket

1. Convert $60°$ to radians.

2. Convert $-\frac{\pi}{2}$ rad to degrees.

3. Explain how radian measure is related to the radius of a circle. Draw and label an appropriate diagram to support your response.

Exit Ticket Sample Solutions

1. **Convert $60°$ to radians.**

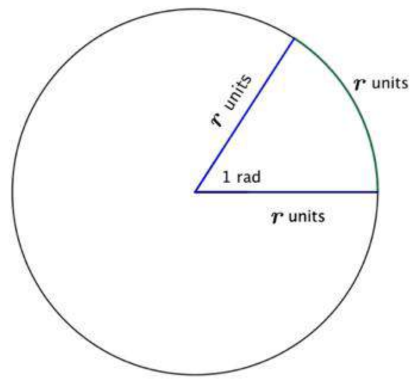
   $60° = \frac{\pi}{3}$ rad

2. **Convert $-\frac{\pi}{2}$ rad to degrees.**

   $-\frac{\pi}{2}$ rad $= -90°$

3. **Explain how radian measure is related to the radius of a circle. Draw and label an appropriate diagram to support your response.**

   If we take one radius from the circle and wrap it part of the way around the circle, the central angle with the arc length of one radius has measure one radian.

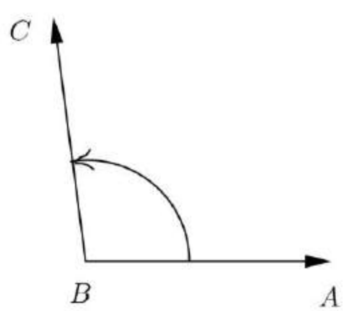


Problem Set Sample Solutions

For Problem 1, students need to have access to a protractor that measures in radians. The majority of the problems in this Problem Set are designed to build fluency with radians and encourage the shift from thinking in terms of degrees to thinking in terms of radians. For Problem 12, ask students to compare the lengths they calculate to the lengths found at http://en.wikipedia.org/wiki/Latitude.

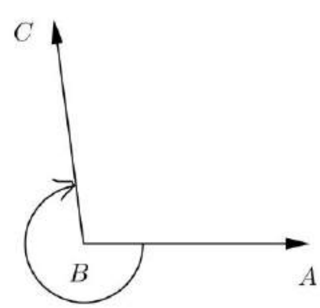
1. **Use a radian protractor to measure the amount of rotation in radians of ray BA to ray BC in the indicated direction. Measure to the nearest 0.1 radian. Use negative measures to indicate clockwise rotation.**

   a.



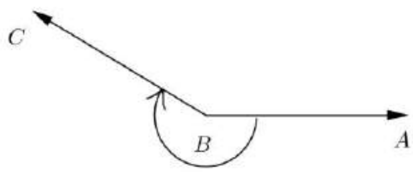
   1.7 rad

   b.



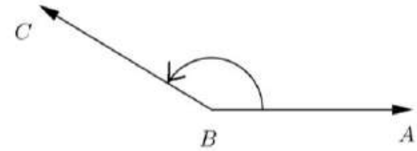
   -4.6 rad

c.



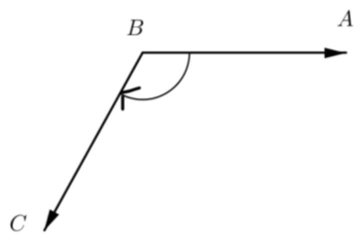
-3.4 rad

d.



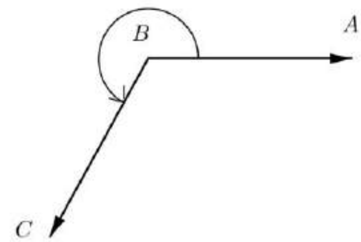
2.6 rad

e.



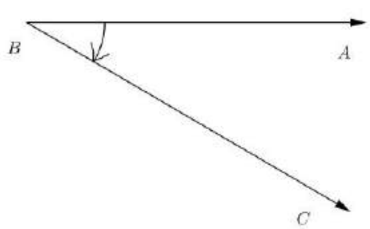
-2.1 rad

f.



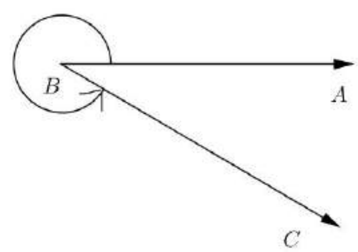
4.2 rad

g.



-0.4 rad

h.



5.8 rad

2. **Complete the table below, converting from degrees to radians. Where appropriate, give your answers in the form of a fraction of π.**

Degrees	Radians
$90°$	$\frac{\pi}{2}$
$300°$	$\frac{5\pi}{3}$
$-45°$	$-\frac{\pi}{4}$
$-315°$	$-\frac{7\pi}{4}$
$-690°$	$-\frac{23\pi}{6}$
$3\frac{3}{4}°$	$\frac{\pi}{48}$
$90\pi°$	$\frac{\pi^2}{2}$
$-\frac{45°}{\pi}$	$-\frac{1}{4}$

3. Complete the table below, converting from radians to degrees.

Radians	Degrees
$\frac{\pi}{4}$	$45°$
$\frac{\pi}{6}$	$30°$
$\frac{5\pi}{12}$	$75°$
$\frac{11\pi}{36}$	$55°$
$-\frac{7\pi}{24}$	$-52.5°$
$-\frac{11\pi}{12}$	$-165°$
49π	$8820°$
$\frac{49\pi}{3}$	$2940°$

4. Use the unit circle diagram from the end of the lesson and your knowledge of the six trigonometric functions to complete the table below. Give your answers in exact form, as either rational numbers or radical expressions.

θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$	$\cot(\theta)$	$\sec(\theta)$	$\csc(\theta)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
0	1	0	0	*undefined*	1	*undefined*
$-\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$-\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
$-\frac{11\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

5. Use the unit circle diagram from the end of the lesson and your knowledge of the sine, cosine, and tangent functions to complete the table below. Select values of θ so that $0 \leq \theta < 2\pi$.

θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
π	-1	0	0
$\frac{3\pi}{2}$	0	-1	*undefined*
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$

6. How many radians does the minute hand of a clock rotate through over 10 minutes? How many degrees?

In 10 minutes, the minute hand makes $\frac{1}{6}$ of a rotation, so it rotates through $\frac{1}{6}(2\pi) = \frac{\pi}{3}$ radians. This is equivalent to $60°$.

7. How many radians does the minute hand of a clock rotate through over half an hour? How many degrees?

In 30 minutes, the minute hand makes $\frac{1}{2}$ of a rotation, so it rotates through $\frac{1}{2}(2\pi) = \pi$ radians. This is equivalent to $180°$.

8. What is the radian measure of an angle subtended by an arc of a circle with radius 4 cm if the intercepted arc has length 14 cm? How many degrees?

The intercepted arc is the length of 3.5 radii, so the angle subtended by that arc measures 3.5 radians. This is equivalent to $3.5\left(\frac{180°}{\pi}\right) = \frac{630°}{\pi} \approx 200.5°$.

9. What is the radian measure of an angle formed by the minute and hour hands of a clock when the clock reads 1:30? How many degrees? (Hint: You must take into account that the hour hand is not directly on the 1.)

At 1:30, the hour hand is halfway between the 1 and the 2, and the minute hand is on the 6. A hand on the clock rotates through $\frac{1}{12}$ of a rotation as it moves from one number to the next. Since there are $4\frac{1}{2}$ of these increments between the two hands of the clock at 1:30, the angle formed by the two clock hands is $\frac{9}{2}\left(\frac{1}{12}\right)(2\pi) = \frac{3\pi}{4}$ radians. In degrees, this is $135°$.

10. **What is the radian measure of an angle formed by the minute and hour hands of a clock when the clock reads 5:45? How many degrees?**

At 5:45, the hour hand is $\frac{3}{4}$ of the way between the 5 and 6 on the clock face, and the minute hand is on the 9. Then there are $3\frac{1}{4}$ increments of $\frac{1}{12}$ of a rotation between the two hands of the clock at 5:45, so the angle formed by the two clock hands is $\left(\frac{13}{4}\right)\left(\frac{1}{12}\right)(2\pi) = \frac{13}{24}\pi$ radians. This is equivalent to $\frac{13\pi}{24}\left(\frac{180°}{\pi}\right) = 97.5°$.

11. **How many degrees does the earth revolve on its axis each hour? How many radians?**

The earth revolves through $360°$ in 24 hours, so it revolves $\frac{360°}{24} = 15°$ each hour.

12. **The distance from the equator to the North Pole is almost exactly $10{,}000$ km.**

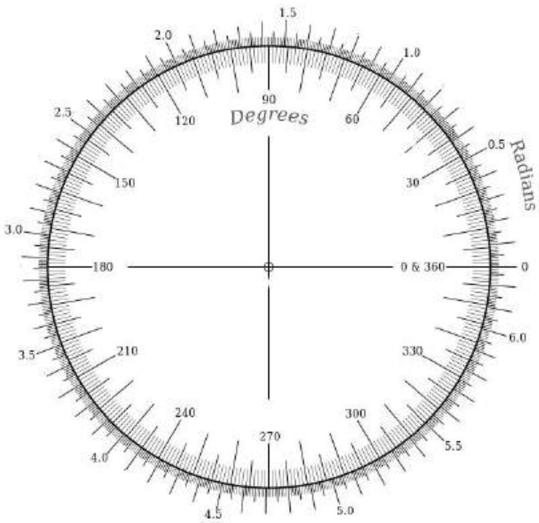
    a. **Roughly how many kilometers is 1 degree of latitude?**

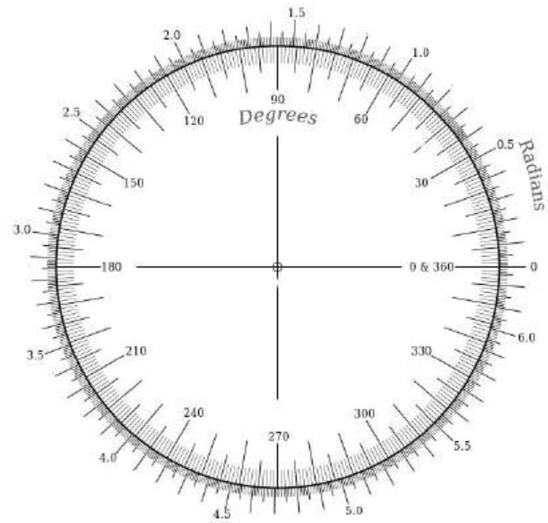
    There are 90 degrees of latitude between the equator and the North Pole, so each degree of latitude is $\frac{1}{90}(10000) \approx 111.1$ km.

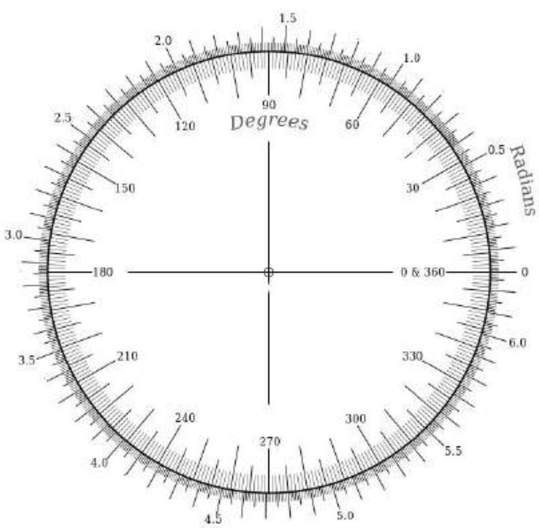
    b. **Roughly how many kilometers is 1 radian of latitude?**

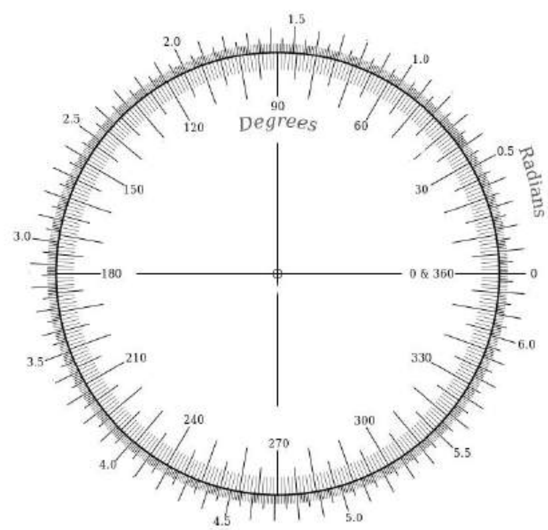
    There are $\frac{\pi}{2}$ radians of latitude between the equator and the North Pole, so each radian of latitude is $\frac{2}{\pi}(10{,}000) \approx 6{,}366.2$ km.

Supplementary Transparency Materials









Lesson 10: Basic Trigonometric Identities from Graphs

Student Outcomes

- Students observe identities from graphs of sine and cosine basic trigonometric identities and relate those identities to periodicity, even and odd properties, intercepts, end behavior, and cosine as a horizontal translation of sine.

Lesson Notes

Students have previous experience with graphing the sine and cosine functions in degrees and have been introduced to radian measure in the previous lesson. For the remainder of the module, students will use radians to work with and graph trigonometric functions. The purpose of this lesson is to increase students' comfort with radians and to formalize the characteristics of periodicity, end behavior, intercepts, and relative extrema of the sine and cosine functions through the observation and conjecture of some basic trigonometric identities. As students work through the lesson, make sure that they grasp the following identities that are valid for all real numbers x:

$$\sin(x+2\pi)=\sin(x) \qquad \sin(x+\pi)=-\sin(x) \qquad \sin(-x)=-\sin(x)$$

$$\cos(x+2\pi)=\cos(x) \qquad \cos(x+\pi)=-\cos(x) \qquad \cos(-x)=\cos(x)$$

and that they know how to use these identities to evaluate sine and cosine for a variety of values of x. Discovering these six trigonometric identities through observation is the focus of this lesson. Discovering and proving other identities will be the focus of Lessons 15, 16, and 17 of this module.

It is important to note that an identity is a statement that two functions are equal on a common domain. Thus, to specify an identity, students need to specify both an equation and a set of values for the variable for which the statement is true. That is, the statement $\sin(-x)=-\sin(x)$ itself is not an identity, but the statement $\sin(-x)=-\sin(x)$ for all real numbers x is an identity.

Materials

Supply students with colored pencils for the activity.

Classwork

Opening (2 minutes)

Students should be able to largely work through these explorations in groups without too much assistance. It may be necessary to help students get started in constructing their table of values by reminding them of the values where the sine and cosine functions take on the values 0, 1, and -1. Also, remind students that they have seen these graphs earlier in the module, but the graphs were constructed using degree measures. Today (and for the remainder of the module) graphing uses radian measures.

Exploratory Challenge 1 (18 minutes)

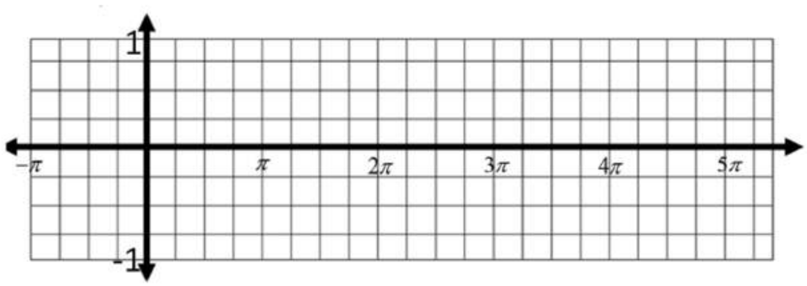
Allow students to work in groups through this problem. Because students are repeating a similar process to discover three different identities, assign groups different parts of the Exploratory Challenge, and then have them report their results to the class. If students are having trouble with the table, help them get started by reminding them that the maximum and minimum values and x-intercepts can be found by evaluating sine at the extreme points on the circle. Circulate around the room to ensure that students are comfortable working with radian values and that they are using the graph to discover the identities. Debrief after this activity to ensure that students have discovered the identities correctly.

Exploratory Challenge 1

Consider the function $f(x) = \sin(x)$ where x is measured in radians.

Graph $f(x) = \sin(x)$ on the interval $[-\pi, 5\pi]$ by constructing a table of values. Include all intercepts, relative maximum points, and relative minimum points of the graph. Then, use the graph to answer the questions that follow.

x	
$f(x)$	



a. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$$x = -\frac{\pi}{2} \text{ and } x = -\frac{\pi}{2} + 2\pi$$

$$x = \pi \text{ and } x = \pi + 2\pi$$

$$x = \frac{7\pi}{4} \text{ and } x = \frac{7\pi}{4} + 2\pi$$

b. What can be said about the y-values for each pair of x-values marked on the graph?

For each pair of x-values, the y-values are the same.

MP.8

c. Will this relationship hold for any two x-values that differ by 2π? Explain how you know.

Yes. Since the sine function repeats every 2π units, then the relationship described in part (b) will hold for any two x-values that differ by 2π.

Scaffolding:

- This could also be accomplished using technology. Students could observe these identities by using the table or graph feature of a graphing calculator.
- Advanced learners could be asked to write similar identities for the cosecant function. For example, since $\sin(x + \pi) = -\sin(x)$, does it follow that $\csc(x + \pi) = -\csc(x)$? Test it either by using test values or by exploring the graph of cosecant.

MP.8

d. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\sin(x + 2\pi) =$ ___________.

$\sin(x + 2\pi) = \sin(x)$

e. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (d) holds for that value of x.

$\sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 2\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$ *(Answers will vary.)*

f. How does the conjecture in part (d) support the claim that the sine function is a periodic function?

The sine function repeats every 2π units because 2π radians is one full turn. Thus, if we rotate the initial ray through $x + 2\pi$ radians, the terminal ray is in the same position as if we had rotated by x radians.

g. Use this identity to evaluate $\sin\left(\frac{13\pi}{6}\right)$.

$\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

h. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$x = -\frac{\pi}{4}$ and $x = -\frac{\pi}{4} + \pi$

$x = 2\pi$ and $x = 2\pi + \pi$

$x = \frac{5\pi}{2}$ and $x = \frac{5\pi}{2} + \pi$

i. What can be said about the y-values for each pair of x-values marked on the graph?

For each pair of x-values, the y-values have the same magnitude but opposite sign.

j. Will this relationship hold for any two x-values that differ by π? Explain how you know.

Yes. Since rotating by an additional π radians produces a point in the opposite quadrant with the same reference angle, the sine of the two numbers will have the same magnitude and opposite sign.

k. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\sin(x + \pi) =$ ___________.

$\sin(x + \pi) = -\sin(x)$

l. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (k) holds for that value of x.

$\sin\left(\frac{3\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \pi\right) = -\sin\left(\frac{\pi}{2}\right) = -1$

m. Is the following statement true or false? Use the conjecture from (k) to explain your answer.

$$\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$$

This statement is true: $\sin\left(\frac{4\pi}{3}\right) = \sin\left(\frac{\pi}{3} + \pi\right) = -\sin\left(\frac{\pi}{3}\right)$.

n. **Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.**

$$x = -\frac{3\pi}{4} \text{ and } x = \frac{3\pi}{4}$$
$$x = -\frac{\pi}{2} \text{ and } x = \frac{\pi}{2}$$

o. **What can be said about the y-values for each pair of x-values marked on the graph?**

For each pair of x-values, the y-values have the same magnitude but with the opposite sign.

p. **Will this relationship hold for any two x-values with the same magnitude but opposite sign? Explain how you know.**

Yes. If rotation by x radians produces the point (a, b) on the unit circle, then rotation by $-x$ radians will produce a point $(a, -b)$ on the unit circle.

q. **Based on these results, make a conjecture by filling in the blank below.**

For any real number x, $\sin(-x) =$ ___________.

$\sin(-x) = -\sin(x)$

r. **Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (q) holds for that value of x.**

For example, $\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so $\sin\left(-\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right)$.

s. **Is the sine function an odd function, even function, or neither? Use the identity from part (q) to explain.**

The sine function is an odd function because $\sin(-x) = -\sin(x)$ and because the graph is symmetric with respect to the origin.

t. **Describe the x-intercepts of the graph of the sine function.**

The graph of the sine function has x-intercepts at all x-values such that $x = n\pi$, where n is an integer.

u. **Describe the end behavior of the sine function.**

As x increases to ∞ or as x decreases to $-\infty$, the sine function cycles between the values of -1 and 1.

During the debriefing, record (or have students record) the key results on the board.

For all x: $\sin(x + 2\pi) = \sin(x)$ $\quad\sin(x + \pi) = -\sin(x)$ $\quad\sin(-x) = -\sin(x)$

Exploratory Challenge 2 (10 minutes)

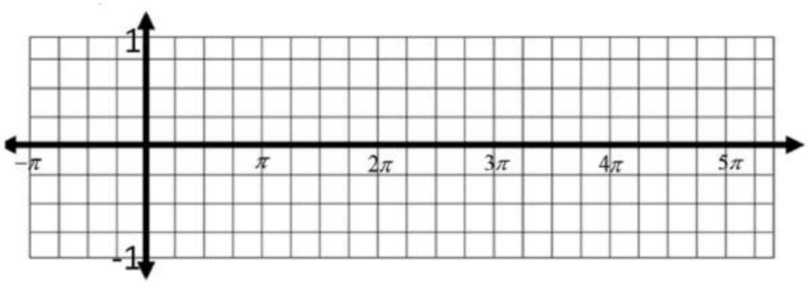
Allow students to work in groups through this problem. This exploration should go more quickly as it is the same process that students went through in Exploratory Challenge 1. As suggested above, assign groups different parts of the Exploratory Challenge, and then have them report their results to the class. Debrief after this activity to ensure that students have completed the identities correctly.

Exploratory Challenge 2

Consider the function $g(x) = \cos(x)$ where x is measured in radians.

Graph $g(x) = \cos(x)$ on the interval $[-\pi, 5\pi]$ by constructing a table of values. Include all intercepts, relative maximum points, and relative minimum points. Then, use the graph to answer the questions that follow.

x	
$g(x)$	



a. **Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.**

$$x = -\frac{\pi}{2} \text{ and } x = -\frac{\pi}{2} + 2\pi$$

$$x = \pi \text{ and } x = \pi + 2\pi$$

$$x = \frac{7\pi}{4} \text{ and } x = \frac{7\pi}{4} + 2\pi$$

b. **What can be said about the y-values for each pair of x-values marked on the graph?**

For each pair of x-values, the y-values are the same.

c. **Will this relationship hold for any two x-values that differ by 2π? Explain how you know.**

Yes. Since the sine function repeats every 2π units, then the relationship in part (b) will hold for any two x-values that differ by 2π.

d. **Based on these results, make a conjecture by filling in the blank below.**

For any real number x, $\cos(x + 2\pi) =$ ___________.

$\cos(x + 2\pi) = \cos(x)$

e. **Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (d) holds for that value of x.**

$\cos(0) = \cos(0 + 2\pi) = 1$ *(Answers will vary.)*

f. **How does the conjecture from part (d) support the claim that the cosine function is a periodic function?**

Like the sine function, the cosine function repeats every 2π units because 2π radians is one full turn. Thus, if we rotate the initial ray through $x + 2\pi$ radians, the terminal ray is in the same position as if we had rotated by x radians.

g. **Use this identity to evaluate $\cos\left(\frac{9\pi}{4}\right)$.**

$\cos\left(\frac{9\pi}{4}\right) = \cos\left(\frac{\pi}{4} + 2\pi\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

h. **Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.**

$$x = -\frac{\pi}{4} \text{ and } x = -\frac{\pi}{4} + \pi$$
$$x = 2\pi \text{ and } x = 2\pi + \pi$$
$$x = \frac{5\pi}{2} \text{ and } x = \frac{5\pi}{2} + \pi$$

i. **What can be said about the y-values for each pair of x-values marked on the graph?**

For each pair of x-values, the y-values have the same magnitude but opposite sign.

j. **Will this relationship hold for any two x-values that differ by π? Explain how you know.**

Yes. Since rotating by an additional π radians produces a point in the opposite quadrant with the same reference angle, the sine of the two numbers will have the same magnitude and opposite sign.

k. **Based on these results, make a conjecture by filling in the blank below.**

For any real number x, $\cos(x + \pi) =$ ___________.

$\cos(x + \pi) = -\cos(x)$

l. **Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (k) holds for that value of x.**

$\cos(3\pi) = \cos(2\pi + \pi) = -\cos(2\pi) = -1$

m. **Is the following statement true or false? Use the identity from part (k) to explain your answer.**

$$\cos\left(\frac{5\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right)$$

This statement is true: $\cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{2\pi}{3} + \pi\right) = -\cos\left(\frac{2\pi}{3}\right)$.

n. **Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.**

$$x = -\frac{3\pi}{4} \text{ and } x = \frac{3\pi}{4}$$
$$x = -\pi \text{ and } x = \pi$$

o. What can be said about the y-values for each pair of x-values marked on the graph?

For each pair of x-values, the y-values have the same magnitude and the same sign.

p. Will this relationship hold for any two x-values with the same magnitude and same sign? Explain how you know.

Yes. If rotation by x radians produces the point (a, b) on the unit circle, then rotation by $-x$ radians will produce a point $(a, -b)$ on the unit circle.

q. Based on these results, make a conjecture by filling in the blank below.

For any real number , $\cos(-x) =$ ____________.

$\cos(-x) = \cos(x)$

r. Test your conjecture by selecting another x-value from the graph and demonstrating that the identity is true for that value of x.

$\cos(-2\pi) = \cos(2\pi) = 1$

s. Is the cosine function an odd function, even function, or neither? Use the identity from part (n) to explain.

The cosine function is an even function because $\cos(-x) = \cos(x)$ and because the graph is symmetric with respect to the y-axis.

t. Describe the x-intercepts of the graph of the cosine function.

The graph of the cosine function has x-intercepts at all x-values such that $x = \frac{\pi}{2} + n\pi$, where n is an integer.

u. Describe the end behavior of $g(x) = \cos(x)$.

As x increases to ∞ or as x decreases to $-\infty$, the cosine function cycles between the values of -1 and 1.

Scaffolding:

Advanced learners could be asked to write similar identities for the tangent function. For example, does $\tan(-x)$ equal $\tan(x)$ or $\tan(-x)$ or neither? Prove it by applying the properties of sine and cosine.

During the debriefing, record (or have students record) the key results on the board.

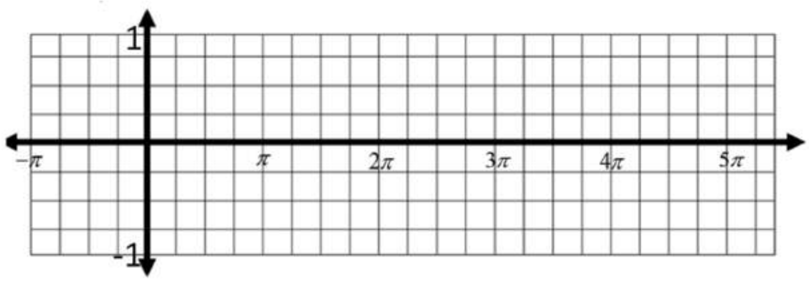
For all x: $\cos(x + 2\pi) = \cos(x)$ $\cos(x + \pi) = -\cos(x)$ $\cos(-x) = \cos(x)$

Exploratory Challenge 3 (8 minutes)

Allow students to work in groups through this problem. To save time, provide students with a paper that already has the two functions graphed together. Debrief after this exercise to ensure that students have completed the identities correctly.

Exploratory Challenge 3

Graph both $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on the graph below. Then, use the graphs to answer the questions that follow.



a. **List ways in which the graphs of the sine and cosine functions are alike.**

Both functions are periodic and have a period of 2π. Both functions have a domain of all real numbers and a range of $[-1, 1]$. Both functions cycle between -1 and 1 as x increases to ∞ or as x decreases to $-\infty$.

Both have similar identities such as $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.

b. **List ways in which the graphs of the sine and cosine functions are different.**

We stated above that $f(x) = \sin(x)$ is an odd function, and $g(x) = \cos(x)$ is an even function. Where the sine function is at a maximum or minimum point, the cosine function has an x-intercept and vice versa.

c. **What type of transformation would be required to make the graph of the sine function coincide with the graph of the cosine function?**

A horizontal shift

d. **What is the smallest possible horizontal translation required to make the graph of $f(x) = \sin(x)$ coincide with the graph of $g(x) = \cos(x)$?**

A horizontal shift $\frac{\pi}{2}$ units to the left

e. **What is the smallest possible horizontal translation required to make the graph of $g(x) = \cos(x)$ coincide with the graph of $f(x) = \sin(x)$?**

A horizontal shift $\frac{\pi}{2}$ units to the right

f. **Use your answers from parts (d) and (e) to fill in the blank below.**

For any real number x, ________ $= \cos\left(x - \frac{\pi}{2}\right)$.

For any real number x, ________ $= \sin\left(x + \frac{\pi}{2}\right)$.

$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$

Note during the debriefing that there are many different horizontal shifts that could be used in order to make the sine function coincide with the cosine function or vice versa. Ask students for other examples.

Closing (2 minutes)

Ask students to explain to a partner or record an explanation on paper. If time permits, ask them to demonstrate using a specific example.

MP.2

- Explain how the identities *For all real numbers* x, $\cos(x + 2\pi) = \cos(x)$ and *For all real numbers* x, $\sin(x + 2\pi) = \sin(x)$ support the idea that both sine and cosine are periodic.
  - *These identities both state that if* 2π *is added to the value of* x*, the value of sine or cosine does not change. This confirms that the functions are periodic and repeat every* 2π *units.*
- How do these identities help us to evaluate sine and cosine for various x-values? For example, how can we use the fact that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ to find $\sin\left(\frac{5\pi}{4}\right)$?
  - $\sin\left(\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4} + \pi\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

Lesson Summary

For all real numbers x:

$\sin(x + 2\pi) = \sin(x)$	$\cos(x + 2\pi) = \cos(x)$
$\sin(x + \pi) = -\sin(x)$	$\cos(x + \pi) = -\cos(x)$
$\sin(-x) = -\sin(x)$	$\cos(-x) = -\cos(x)$
$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$	$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 10: Basic Trigonometric Identities from Graphs

Exit Ticket

1. Demonstrate how to evaluate $\cos\left(\frac{8\pi}{3}\right)$ by using a trigonometric identity.

2. Determine if the following statement is true or false, without using a calculator.

$$\sin\left(\frac{8\pi}{7}\right) = \sin\left(\frac{\pi}{7}\right)$$

3. If the graph of the cosine function is translated to the right $\frac{\pi}{2}$ units, the resulting graph is that of the sine function, which leads to the identity: For all x, $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$. Write another identity for $\sin(x)$ using a different horizontal shift.

Exit Ticket Sample Solutions

1. **Demonstrate how to evaluate $\cos\left(\frac{8\pi}{3}\right)$ by using a trigonometric identity.**

   $\cos\left(\frac{8\pi}{3}\right) = \cos\left(\frac{2\pi}{3} + 2\pi\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

2. **Determine if the following statement is trueor false, without using a calculator.**

   $$\sin\left(\frac{8\pi}{7}\right) = \sin\left(\frac{\pi}{7}\right)$$

   False. $\sin\left(\frac{8\pi}{7}\right) = \sin\left(\frac{\pi}{7} + \pi\right) = -\sin\left(\frac{\pi}{7}\right) \neq \sin\left(\frac{\pi}{7}\right)$

3. **If the graph of the cosine function is translated to the right $\frac{\pi}{2}$ units, the resulting graph is that of the sine function, which leads to the identity: For all x, $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$. Write another identity for $\sin(x)$ using a different horizontal shift.**

   $\sin(x) = \cos\left(x + \frac{3\pi}{2}\right)$

Problem Set Sample Solutions

1. **Describe the values of x for which each of the following is true.**
   a. **The cosine function has a relative maximum.**

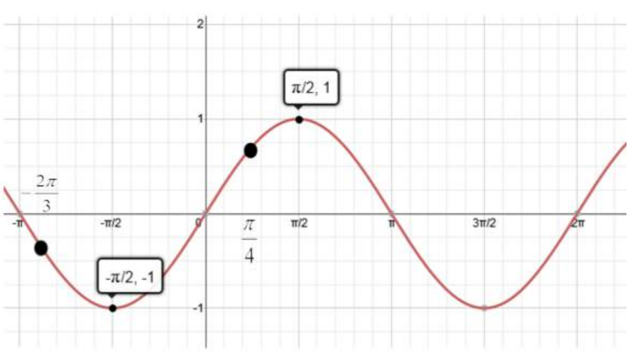
      The cosine function has a relative maximum at all $x = 2\pi n$, where n is an integer.

   b. **The sine function has a relative maximum.**

      The sine function has a relative maximum at all $x = \frac{\pi}{2} + 2\pi n$, where n is an integer.

2. **Without using a calculator, rewrite each of the following in order from least to greatest. Use the graph to explain your reasoning.**

   $\sin\left(\frac{\pi}{2}\right)$ $\quad\sin\left(-\frac{2\pi}{3}\right)$ $\quad\sin\left(\frac{\pi}{4}\right)$ $\quad\sin\left(-\frac{\pi}{2}\right)$

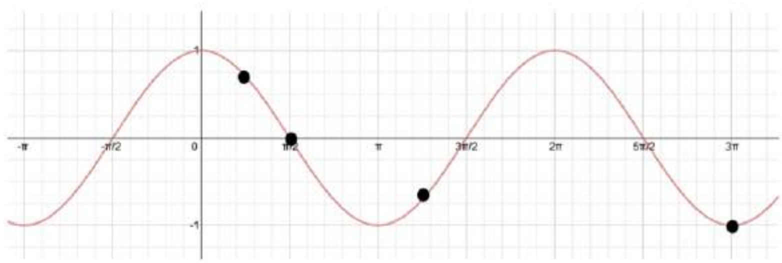


At $-\frac{\pi}{2}$, the sine function takes on its smallest value, and at $\frac{\pi}{2}$ the sine function takes on its largest value. Additionally, $\sin\left(-\frac{2\pi}{3}\right)$ is negative, and $\sin\left(\frac{\pi}{4}\right)$ is positive. Then, these four values in increasing order are

$\sin\left(-\frac{\pi}{2}\right)$, $\sin\left(-\frac{2\pi}{3}\right)$, $\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{\pi}{2}\right)$.

3. **Without using a calculator, rewrite each of the following in order from least to greatest. Use the graph to explain your reasoning.**

$\cos\left(\frac{\pi}{2}\right)$ $\cos\left(\frac{5\pi}{4}\right)$ $\cos\left(\frac{\pi}{4}\right)$ $\cos(5\pi)$



At 5π, the cosine function takes on its smallest value because $5\pi = 2\pi + 2\pi + \pi$, so $\cos(5\pi) = \cos(\pi) = -1$. We can see that $\cos\left(\frac{5\pi}{4}\right)$ is negative, $\cos\left(\frac{\pi}{2}\right) = 0$, and $\cos\left(\frac{\pi}{4}\right)$ is positive. Then, these four values in increasing order are

$$\cos(5\pi), \cos\left(\frac{5\pi}{4}\right), \cos\left(\frac{\pi}{2}\right), \cos\left(\frac{\pi}{4}\right).$$

4. **Evaluate each of the following without a calculator using a trigonometric identity when needed.**

$\cos\left(\frac{\pi}{6}\right)$ $\cos\left(-\frac{\pi}{6}\right)$ $\cos\left(\frac{7\pi}{6}\right)$ $\cos\left(\frac{13\pi}{6}\right)$

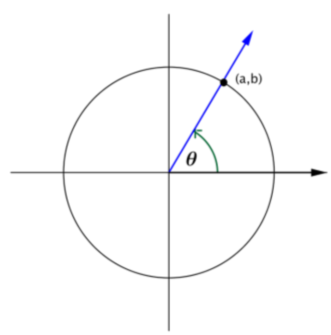
$\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{2}$ $-\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{2}$

5. **Evaluate each of the following without a calculator using a trigonometric identity when needed.**

$\sin\left(\frac{3\pi}{4}\right)$ $\sin\left(\frac{11\pi}{4}\right)$ $\sin\left(\frac{7\pi}{4}\right)$ $\sin\left(\frac{-5\pi}{4}\right)$

$\frac{\sqrt{2}}{2}$ $\frac{\sqrt{2}}{2}$ $-\frac{\sqrt{2}}{2}$ $\frac{\sqrt{2}}{2}$

6. **Use the rotation through θ radians shown to answer each of the following questions.**



a. **Explain why $\sin(-\theta) = -\sin(\theta)$ for all real numbers θ.**

If the initial ray is rotated through $-\theta$ radians, it is rotated θ radians in the clockwise direction. If the terminal ray of a rotation through θ radians intersects the unit circle at the point (a, b), then the terminal ray of rotation through $-\theta$ radians will intersect the unit circle at the point $(a, -b)$. Then, $\sin(\theta) = b$ and $\sin(-\theta) = -b$, so $\sin(-\theta) = -\sin(\theta)$.

b. **What symmetry does this identity demonstrate about the graph of $y = \sin(x)$?**

It demonstrates that the graph of $y = \sin(x)$ is symmetric with respect to the origin because if the point (x, y) is on the graph, then the point $(-x, -y)$ is also on the graph.

7. **Use the same rotation shown in Problem 6 to answer each of the following questions.**

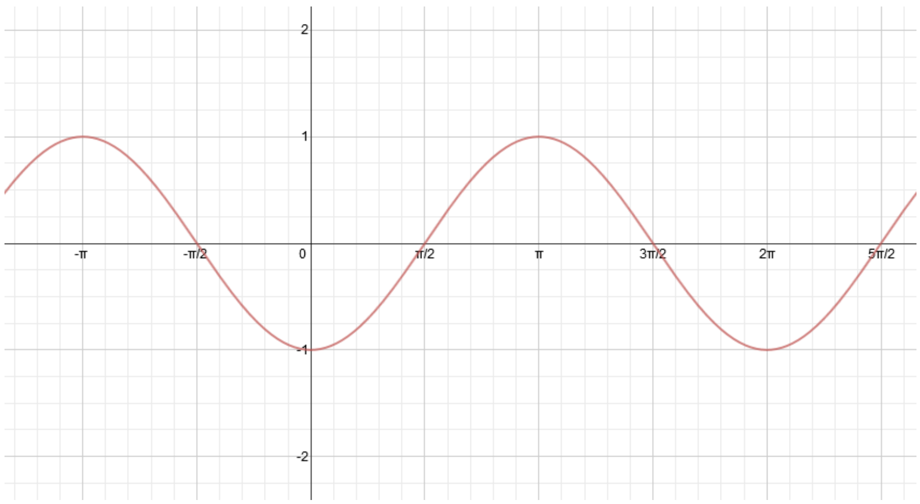
   a. **Explain why $\cos(-\theta) = \cos(\theta)$.**

   If the initial ray is rotated through $-\theta$ radians, it is rotated θ radians in the clockwise direction. If the terminal ray of a rotation through θ radians intersects the unit circle at the point (a, b), then the terminal ray of rotation through $-\theta$ radians will intersect the unit circle at the point $(a, -b)$. Then, $\cos(\theta) = a$ and $\cos(-\theta) = a$, so $\cos(-\theta) = \cos(\theta)$.

   b. **What symmetry does this identity demonstrate about the graph of $y = \cos(x)$?**

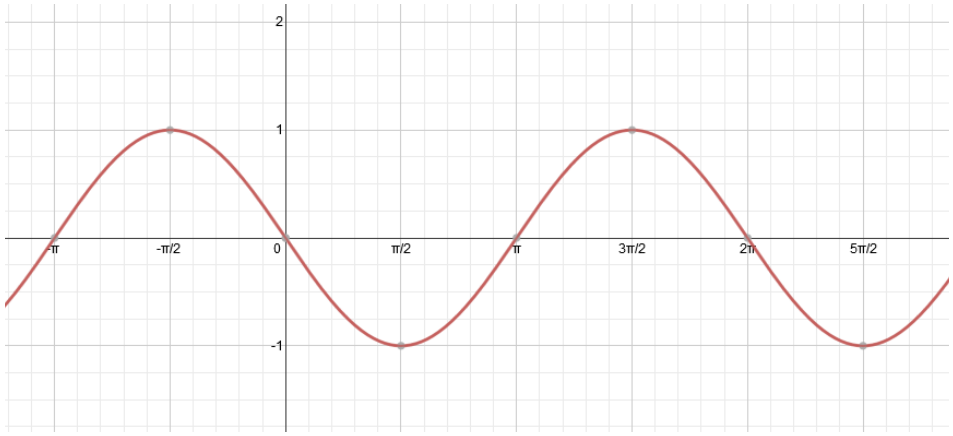
   It demonstrates that the graph of $y = \cos(x)$ is symmetric with respect to the y-axis because if the point (x, y) is on the graph, then the point $(-x, y)$ is also on the graph.

8. **Find equations of two different functions that can be represented by the graph shown below—one sine and one cosine—using different horizontal transformations.**



$y = \sin\left(x - \frac{\pi}{2}\right)$ $y = \cos(x - \pi)$ **(Other correct answers are possible.)**

9. **Find equations of two different functions that can be represented by the graph shown below—one sine and one cosine—using different horizontal translations.**



$y = \cos\left(x + \frac{\pi}{2}\right)$ $y = \sin(x - \pi)$ **(Other correct answers are possible.)**

Lesson 11: Transforming the Graph of the Sine Function

Student Outcomes

- Students formalize the period, frequency, phase shift, midline, and amplitude of a general sinusoidal function by understanding how the parameters A, ω, h, and k in the formula
$$f(x) = A\sin\big(\omega(x-h)\big) + k$$
are used to transform the graph of the sine function and how variations in these constants change the shape and position of the graph of the sine function.
- Students learn the relationship among the constants A, ω, h, and k in the formula $f(x) = A\sin\big(\omega(x-h)\big) + k$ and the properties of the sine graph. In particular, they learn that:
  - $|A|$ is the *amplitude* of the function. The amplitude is the distance between a maximal point of the graph of the sinusoidal function and the midline (i.e., $A = f_{max} - k$ or $A = \frac{f_{max}-f_{min}}{2}$).
  - $\frac{2\pi}{|\omega|}$ is the *period* of the function. The period P is the distance between two consecutive maximal points (or two consecutive minimal points) on the graph of the sinusoidal function. Thus, $\omega = \frac{2\pi}{P}$.
  - $\frac{|\omega|}{2\pi}$ is the *frequency* of the function (the frequency is the reciprocal of the period).
  - h is called the *phase shift*.
  - The graph of $y = k$ is called the *midline*.
  - Furthermore, the graph of the sinusoidal function f is obtained by vertically scaling the graph of the sine function by A, horizontally scaling the resulting graph by $\frac{1}{\omega}$, and then horizontally and vertically translating the resulting graph by h and k units, respectively.

Lesson Notes

The lesson is planned for one day, but the teacher may choose to extend it to two days. The Ferris wheel exploration in Lesson 12 continues to expand on the ideas introduced in this lesson, as do the modeling problems in Lesson 13. This lesson explores the effects of changing the parameters on the graphs of sinusoidal functions, and the next lessons work backward to write a formula for sinusoidal function given its graph and to fit a sinusoidal function to data. Students work in groups to discover the effects of changing each of the parameters A, ω, h, and k in the generalized sine function $f(x) = A\sin\big(\omega(x-h)\big) + k$. Note that the character ω is a lowercase Greek *omega*, not a "w". The parameters h and k in the formula $f(x) = A\sin\big(\omega(x-h)\big) + k$ play a similar role as the h and k in the vertex form $p(x) = a(x-h)^2 + k$ of a quadratic function.

This lesson draws heavily on MP.7 and MP.8. Students graph functions repeatedly (MP.8) to generalize the effect of the parameters (MP.7) on the graph. During this lesson, students are grouped twice. First, they are arranged into the four teams that discover the effects of each parameter on the graph of the sine function, and then those teams are scrambled to create new groups that contain at least one member of each original team. It might be a good idea to carefully plan these groups in advance so that each time the groups are scrambled, they contain students at different levels of ability.

The following background information provides a formal definition of terms associated with periodic functions and how the features of the graph of a sinusoidal function relate to the parameters in the generalized function $f(x) = A\sin\big(\omega(x-h)\big) + k$.

PERIODIC FUNCTION: A function f whose domain is a subset of the real numbers is said to be *periodic with period* $P > 0$ if the domain of f contains $x + P$ whenever it contains x and if $f(x+P) = f(x)$ for all real numbers x in its domain.

If a least positive number P exists that satisfies this equation, it is called the *fundamental period*, or if the context is clear, just the *period* of the function.

CYCLE: Given a periodic function, a *cycle* is any closed interval of the real numbers of length equal to the period of the function.

When a periodic function or the graph of a periodic function is said to *repeat,* it means that the graph of the function over a cycle is congruent to the graph of the periodic function over any translation of that cycle to the left or right by multiples of the period. Because a periodic function repeats predictably, one can ask how many repetitions of the function occur in an interval of a given length. The answer is a proportional relationship between lengths of intervals and the cycles in them (thought of as subintervals that intersect only at their endpoints). For example, there are 8 cycles of the function $g(x) = \sin(8x)$ occurring in the interval $[0, 2\pi]$, which can be written as 8 cycles per 2π units, or as the rate $\frac{8}{2\pi}$ cycles/unit. The unit rate of this rate, $\frac{8}{2\pi}$, is called the frequency.

FREQUENCY: The *frequency* of a periodic function is the unit rate of the constant rate defined by the number of cycles per unit length.

In practice, the frequency is often calculated by choosing an interval of a convenient length. For example, we can easily calculate the period and frequency for the function $g(x) = \sin(\omega x)$ from the definitions above in terms of ω. Let P be the period of the function and f its frequency. Since the period of the sine function is 2π, the period of g is $\frac{2\pi}{|\omega|}$ (because the graph of g is a horizontal scaling of the graph of the sine function by $\frac{1}{\omega}$). Thus, $P = \frac{2\pi}{|\omega|}$. Since the length of any cycle is $\frac{2\pi}{|\omega|}$, there are clearly $|\omega|$ cycles in the interval $[0, 2\pi]$. The relationship of $|\omega|$ cycles per 2π units can be used to calculate the number of cycles in any interval of any length; therefore, the proportional relationship can be written as the rate $\frac{|\omega|}{2\pi}$ cycles/unit. The frequency is then $f = \frac{|\omega|}{2\pi}$. From the equations $P = \frac{2\pi}{|\omega|}$ and $f = \frac{|\omega|}{2\pi}$, one can see that period and frequency are reciprocals of each other; that is, $P = \frac{1}{f}$.

Sinusoidal functions are useful for modeling simple harmonic motion.

SINUSOIDAL FUNCTION: A periodic function $f\colon \mathbb{R} \to \mathbb{R}$ is *sinusoidal* if it can be written in the form $f(x) = A\sin(\omega(x-h)) + k$ for real numbers A, ω, h, and k.

In this form,

- $|A|$ is called the *amplitude* of the function.
- $P = \frac{2\pi}{|\omega|}$ is the *period* of the function, and $f = \frac{|\omega|}{2\pi}$ is the *frequency* of the function (the frequency is the reciprocal of the period).
- h is called the *phase shift*, and the graph of $y = k$ is called the *midline*.

Furthermore, one can see that the graph of the sinusoidal function f is obtained by vertically scaling the graph of the sine function by A, horizontally scaling the resulting graph by $\frac{1}{\omega}$, and then horizontally and vertically translating the resulting graph by h and k units, respectively.

To determine the amplitude, period, phase shift, and midline from a graph of a sinusoidal function (or data given as ordered pairs), let f_{max} be the maximum value of the function and f_{min} be the minimum value of the function (f has global maximums and minimums because the values of the sine function are bounded between -1 and 1). A sinusoidal function oscillates between two horizontal lines, the minimal line given by the graph of $y = f_{min}$, and the maximal line given by the graph of $y = f_{max}$. Then:

- The midline is the horizontal line that is halfway between the maximal line and the minimal line (i.e., it is the graph of the equation $y = \frac{f_{max}+f_{min}}{2}$. Thus, the value of k can be found by $k = \frac{f_{max}+f_{min}}{2}$).
- The amplitude is the distance between a maximal point of the graph of the sinusoidal function and the midline (i.e., $|A| = f_{max} - k$). Because the midline is halfway between the maximal and minimal values of f, it is also true that $|A| = \frac{f_{max}-f_{min}}{2}$.
- The period P is the distance between two consecutive maximal points (or two consecutive minimal points) on the graph of the sinusoidal function. Thus, the parameter ω is given by $\omega = \frac{2\pi}{P}$.

Note that the process outlined above for determining the parameters in a sinusoidal function guarantees that both A and ω are positive. Vertical and horizontal translations can then be used to fit the graph of the sinusoidal function to the data.

Many sources provide the following form of the equation for a generalized sine function: $f(x) = A\sin(Bx + C) + D$. This form is equivalent to the one introduced in this lesson but is not as useful for graphing the function.

Graphing calculators or another graphing utility are required for the Opening Exercise but should not be allowed for the exercises near the end of the lesson. Remind students to set the calculator to radian mode.

Scaffolding:

- Be sure that each team has a diverse collection of students with different talents. It might be a good idea to have several stronger students on the ω team since this is typically the most difficult parameter for students to understand.
- If a team is having trouble, give a hint about what the team should observe. For example, ask a struggling team what the values for A, ω, h, and k are for the function
$$f(x) = \sin(2x - \pi) + 5.$$
Then, ask students from the various teams to make conjectures about how changing the parameter values changes the appearance of the graphs of the function.

Classwork

Opening Exercise (15 minutes)

Remind students what the graph of the function $f(x) = \sin(x)$ looks like; then, post the following equation of the generalized sine function:

$$f(x) = A\sin\big(\omega(x - h)\big) + k.$$

This formula defines a family of functions that have the same properties but different graphs for different choices of the parameters A, ω, h, and k. The parameters remain constant for a particular function, so they are not variables, but their values can change from one function to another. That is, one function that has this form is $f(x) = 3\sin\big(2(x - \pi)\big) + 6$, and another is $g(x) = 100\sin\big(1.472(x - 0.0024)\big) - 17$.

Students have seen parameters in other equations for graphs; for example, the equation $y = mx + b$ describes a line with parameters m and b that represent the slope and y-intercept of the line. As another example, recall what the parameters a, h, and k tell us about the graph of a quadratic function in the form $p(x) = a(x-h)^2 + k$.

MP.7 & MP.8

Divide the class into four teams called the A-team, the h-team, the k-team, and the ω-team. Each team is responsible for discovering the effect of changing the group's parameter on the graph of the sine function by graphing the function for several different values of the parameter using a graphing calculator or other graphing utility. Be sure that students try positive and negative values for the parameter as well as values that are close to zero. Each student on each team is responsible for teaching students from the other teams what effects changing the team's parameter has on the basic graph of $f(x) = \sin(x)$, so every student needs to fully participate and understand the group's conclusions. Students should keep a graph of $f(x) = \sin(x)$ entered into their calculators and graph a second function where they change their assigned parameter. For the parameter h, encourage teams to work with multiples of fractions of π before moving on to rational numbers. That way, it is easier for them to identify the horizontal scaling and translations compared to the graph of $f(x) = \sin(x)$.

- The A-team experiments by changing the parameter A in the function $f(x) = A\sin(x)$. Examine how different values for A change the graph of the sine function by using a graphing calculator to produce a graph of f. Recommended starting values for team A are

  $$A \in \left\{2, 3, 10, 0, -1, -2, \frac{1}{2}, \frac{1}{5}, -\frac{1}{3}\right\}.$$

  All group members should be prepared to report to the other groups their conclusions about how amplitude is affected by different values of A.

- The h-team experiments by changing the parameter h in the function $f(x) = \sin(x-h)$. Examine how different values for h change the graph of the sine function by using a graphing calculator to produce a graph of f. Recommended starting values for team h are

  $$h \in \left\{\pi, -\pi, \frac{\pi}{2}, -\frac{\pi}{4}, 2\pi, 2, 0, -1, -2, 5, -5\right\}.$$

  All group members should be prepared to report to the other groups their conclusions about how the graph of the function is affected by different values of h.

- The k-team experiments by changing the parameter k in the function $f(x) = \sin(x) + k$. Examine how different values for k change the graph of the sine function by using a graphing calculator to produce a graph of f. Recommended starting values for team k are

  $$k \in \left\{2, 3, 10, 0, -1, -2, \frac{1}{2}, \frac{1}{5}, -\frac{1}{3}\right\}.$$

  All group members should be prepared to report to the other groups their conclusions about how the graph of the function is affected by different values of k.

- The ω-team experiments by changing the parameter ω in the function $f(x) = \sin(\omega x)$. Examine how different values for ω change the graph of the sine function by using a graphing calculator to produce a graph of f. Recommended starting values for team ω are

  $$\omega \in \left\{2,3,5, \frac{1}{2}, \frac{1}{4}, 0, -1, -2, \pi, 2\pi, 3\pi, \frac{\pi}{2}, \frac{\pi}{4}\right\}.$$

  All group members should be prepared to report to the other groups their conclusions about how the graph of the function is affected by different values of ω.

Sample student responses are included below.

Opening Exercise

Explore your assigned parameter in the sinusoidal function $f(x) = A\sin\big(\omega(x-h)\big) + k$. Select several different values for your assigned parameter, and explore the effects of changing the parameter's value on the graph of the function compared to the graph of $f(x) = \sin(x)$. Record your observations in the table below. Include written descriptions and sketches of graphs.

A-Team	ω-Team
$f(x) = A\sin(x)$	$f(x) = \sin(\omega x)$
Suggested A values: $2, 3, 10, 0, -1, -2, \frac{1}{2}, \frac{1}{5}, -\frac{1}{3}$	Suggested ω values: $2, 3, 5, \frac{1}{2}, \frac{1}{4}, 0, -1, -2, \pi, 2\pi, 3\pi, \frac{\pi}{2}, \frac{\pi}{4}$
The graph is a vertical scaling of the graph of $f(x) = \sin(x)$ by a factor of A.	*The graph is a horizontal scaling of the graph of $f(x) = \sin(x)$ by a scale factor of $\frac{1}{\omega}$.*
When $A > 1$, the graph is a vertical stretch of the graph of the sine function.	*When $\omega > 1$, the graph is a horizontal compression of the graph of the sine function.*
When $0 < A < 1$, the graph is a vertical compression of the graph of the sine function.	*When $0 < \omega < 1$, the graph is a horizontal stretch of the graph of the sine function.*
When A is negative, the graph is also a reflection about the horizontal axis of the sine function.	*When ω is negative, the graph is also a reflection about the vertical axis of the sine function.*
When $A = 1$, the graph is the graph of the sine function.	*When $\omega = 1$, the graph is the graph of the sine function.*
Changing A changes the distance between the maximum and minimum points on the graph of the function.	*Changing ω changes the length of the period of the graph of the function.*
The maximum and minimum values of the graph of $f(x) = A\sin(x)$ are A and $-A$ when $A \neq 0$. The value $\lvert A\rvert$ is the distance between the maximum point and the horizontal axis or half the distance between the maximum and minimum points. If $A < 0$, the graph is reflected across the horizontal axis.	*The number $\lvert\omega\rvert$ counts the number of cycles of the graph in the interval $[0, 2\pi]$. If ω is a nonnegative integer, the graph will repeat ω times on the closed interval $[0, 2\pi]$. The length of the period of this function is $\frac{2\pi}{\lvert\omega\rvert}$.*
If $A = 0$, then the function is constant, not sinusoidal, and the graph is the same as the graph of the line $y = 0$.	*If $\omega = 0$, then the function is constant, not sinusoidal, and the graph is the same as the graph of the line $y = 0$.*

k-Team	**h-Team**
$f(x) = \sin(x) + k$	$f(x) = \sin(x - h)$
Suggested k values: $2, 3, 10, 0, -1, -2, \frac{1}{2}, \frac{1}{5}, -\frac{1}{3}$	Suggested h values: $\pi, -\pi, \frac{\pi}{2}, -\frac{\pi}{4}, 2\pi, 2, 0, -1, -2, 5, -5$
The value of k controls the vertical translation of the graph of f compared to the graph of the sine function.	*The value of h controls the horizontal translation of the graph of f compared to the graph of the sine function.*
The graph of f is the graph of the sine function translated vertically by k units.	*The graph of f is the graph of the sine function translated horizontally by h units.*
If $k > 0$, then the graph is translated in the positive direction (to the right) compared to the graph of the sine function.	*If $h > 0$, then the graph is translated in the positive direction (to the right) compared to the graph of the sine function.*
If $k < 0$, then the graph is translated in the negative direction (to the left) compared to the graph of the sine function.	*If $h < 0$, then the graph is translated in the negative direction (to the left) compared to the graph of the sine function.*
If $k = 0$, then the graph is not translated when compared to the graph of the sine function.	*If $h = 0$, then the graph is not translated when compared to the graph of the sine function.*

After the different teams have had an opportunity to explore their assigned parameter, regroup students so that each new team has at least one student who is an "expert" on each parameter. Each "expert" should share the results regarding the originally assigned parameter within the new group, while the other group members take notes on the graphic organizer. When this activity is completed, all students should have notes, including written descriptions and sketches, recorded on their graphic organizers for each of the four parameters.

Discussion (6 minutes)

To ensure that students have recorded accurate information and to transition to the vocabulary of sinusoidal functions that is the focus on this lesson, lead a short discussion, making sure to define the terms below for the entire class. Have one or two student volunteers describe their findings on the effect of changing their parameter on the graph of the basic sine function. Students do not likely know the right mathematical terms for amplitude, horizontal shift, midline, and period, so, after the students present their conclusions, reiterate each group's results using the correct vocabulary. Students should take notes on these terms. A precise definition of each term can be found in the Lesson Notes.

Conclude the discussion with the following information so that all students begin to associate the vocabulary with features of the graphs of sinusoidal functions.

- The *amplitude* is $|A|$. Use the absolute value of A since amplitude is a length. In terms of the maximum value of the function, f_{max}, and the minimum value of the function, f_{min}, the amplitude is given by the formula:
$$|A| = \frac{f_{max} - f_{min}}{2}.$$
- *The phase shift is* h. The graph of $f(x) = A\sin\big(\omega(x - h)\big) + k$ is a horizontal translation of the graph of the sine function by h units. When a sinusoidal function is written in the form $f(x) = A\sin(Bx + C) + D$, the phase shift is the solution to the equation $Bx + C = 0$. This expression $Bx + C$ can also be rewritten as $B\left(x - \left(-\frac{C}{B}\right)\right)$, where $B = \omega$, and $-\frac{C}{B} = h$ in the general sinusoidal function.

- The graph of $y = k$ is the *midline*. The graph of $f(x) = A\sin(\omega(x-h)) + k$ is a vertical translation of the graph of the sine function by k units. In terms of the maximum value of the function, f_{max}, and the minimum value of the function, f_{min}, the value of k is given by the formula
$$k = \frac{f_{max} + f_{min}}{2}.$$
- The *period* is $P = \frac{2\pi}{|\omega|}$. This period is the horizontal distance between two consecutive maximal points of the graph of f (or two consecutive minimal points).
- The *frequency* describes the number of cycles of the graph in the interval $[0, 2\pi]$. The frequency f is related to ω by $f = \frac{|\omega|}{2\pi} = \frac{1}{P}$.

Help students notice that the period is inversely proportional to the value of ω. Students should notice that the frequency and period of any sinusoidal function are reciprocals. In general, for any periodic function, the period P is the smallest positive number P for which $f(x) = f(x+P)$ for all x.

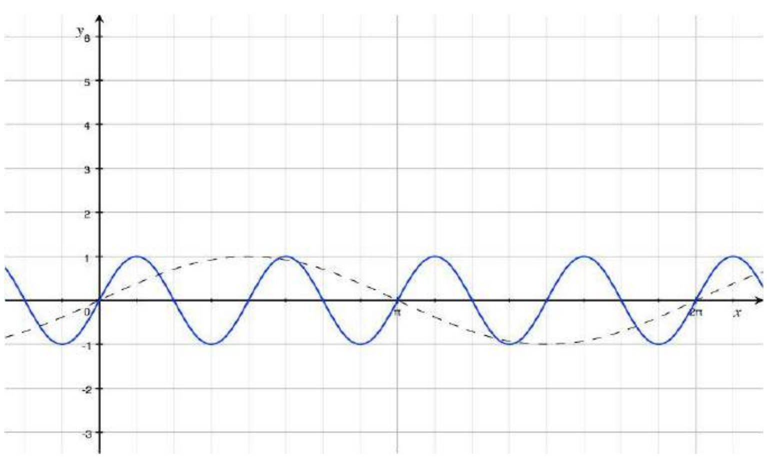
Example (8 minutes)

In this example, walk through a series of four graphs, changing one parameter at a time, to create the final graph of $f(x) = 3\sin\left(4\left(x - \frac{\pi}{6}\right)\right) + 2$. An ordered progression that works well is ω, h, A, k. Ask representatives from the ω-team to describe the change brought by $\omega = 4$, representatives from the h-team to describe the change brought by $h = \frac{\pi}{6}$, and so on, for all four parameters. At each step, show the graph from the previous step together with the new graph so that students can see the change brought about by each of the parameters. If terms such as *cycle*, *period*, and *frequency* have not been discussed with the class, this would be an appropriate point to introduce those terms.

Example

Graph the following function:

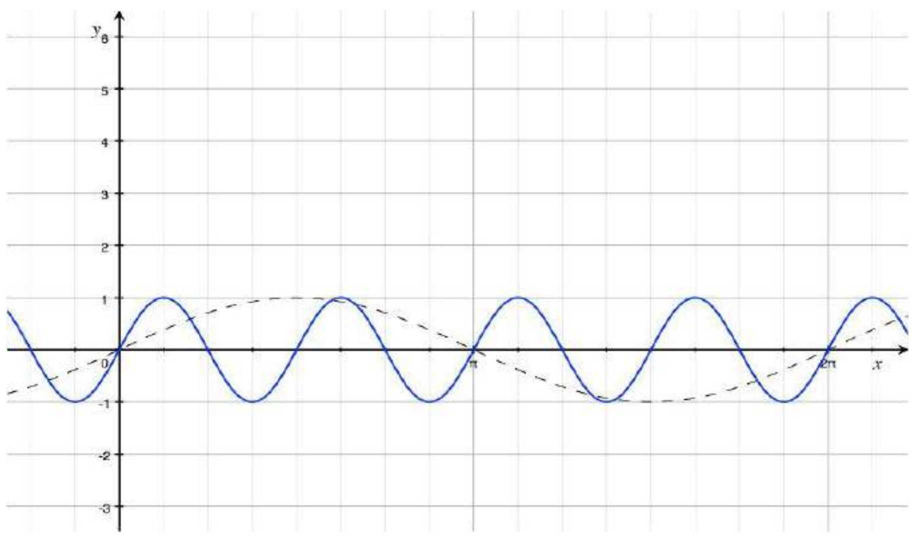
$$f(x) = 3\sin\left(4\left(x - \frac{\pi}{6}\right)\right) + 2.$$



- What are the values of the parameters ω, h, A, and k, and what do they mean?
  - *$\omega = 4$, $h = \frac{\pi}{6}$, $A = 3$, and $k = 2$.*
  - *The value of ω affects the period. The period is $P = \frac{2\pi}{|\omega|} = \frac{\pi}{2}$. The graph of this function is a horizontal scaling of the graph of the sine function by a factor of $\frac{1}{4}$. The graph of this function has four cycles in the closed interval $[0, 2\pi]$.*
  - *The phase shift is $\frac{\pi}{6}$. After shifting the period, we translate the graph $\frac{\pi}{6}$ units to the right.*
  - *The amplitude is 3. The graph of this function is a vertical scaling of the graph of the sine function by a factor of 3. The vertical distance between the maximal and minimal points on the graph of this function is 6 units.*
  - *The graph of the equation $y = 2$ is the midline. We translate the graph we have constructed so far 2 units upward.*

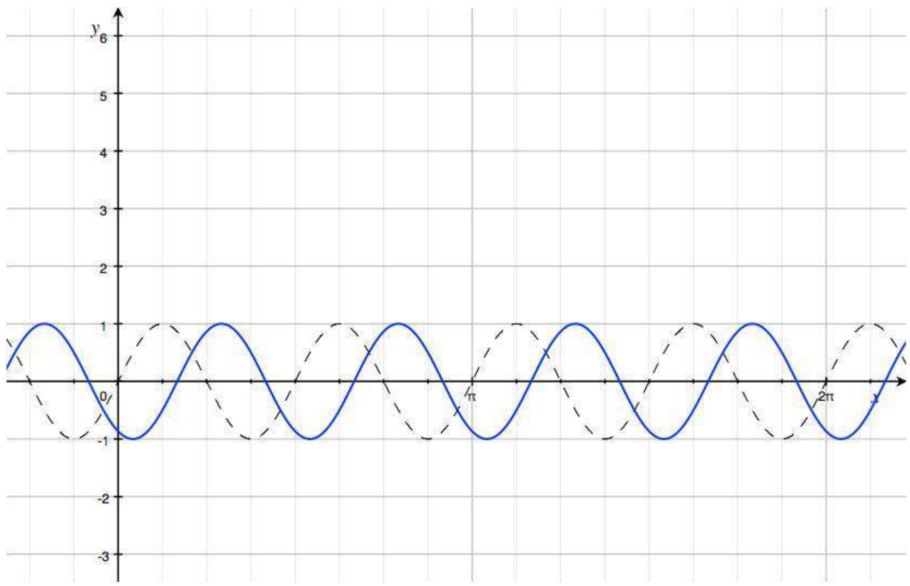
Model the effects of each parameter compared to the graph of the sine function by sketching by hand, using a graphing calculator, or using other graphing software. If, when questioned, students are still using colloquial phrases such as “the graph will squish inward,” then take time to correct their terminology as modeled in the answers below.

- First, we consider the parameter ω, which affects both period and frequency. What happens when $\omega = 4$?
  - *The period is smaller than the period of the sine function. The graph is a horizontal scaling of the original graph by a scale factor of $\frac{1}{4}$.*



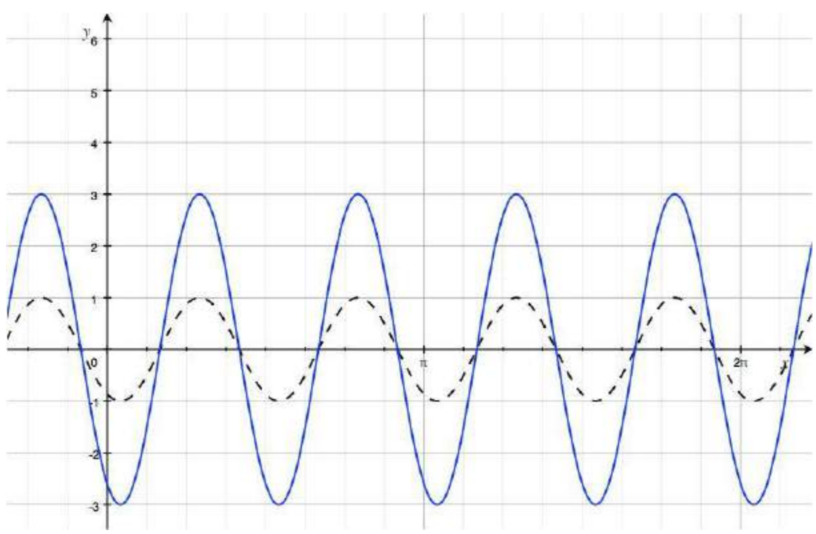
The blue curve is the graph of $y = \sin(4x)$. Be sure to point out the length of the period and that there are four cycles of the graph on the interval $[0, 2\pi]$ because $\omega = 4$.

- Next, we examine the horizontal translation by $h = \frac{\pi}{6}$. What effect does that have on the graph?
  - *The new graph is the old graph shifted* $\frac{\pi}{6}$ *units to the right.*



The blue curve is the graph of $y = \sin\left(4\left(x - \frac{\pi}{6}\right)\right)$. Notice that the point that used to be $(0, 0)$ has now been shifted to $\left(\frac{\pi}{6}, 0\right)$.

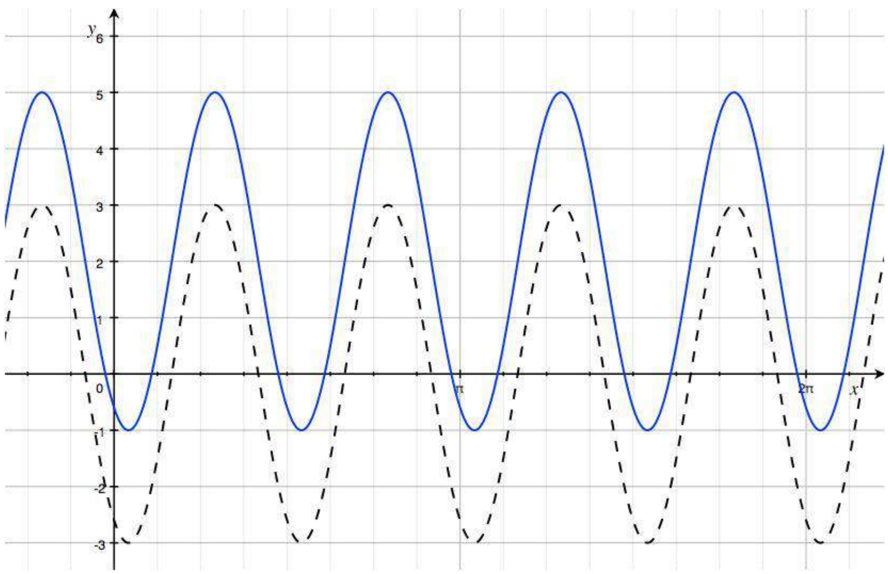
- The next step is to look at the effect of $A = 3$. What does that do to our graph? How does it compare to the previous graph?
  - *The amplitude is* 3*, so the distance between the maximum points and the minimum points is* 6 *units. The graph of this function is the graph of the previous function scaled vertically by a factor of* 3*.*



The blue curve is the graph of $y = 3\sin\left(4\left(x - \frac{\pi}{6}\right)\right)$.

MP.3

- What happens to the zeros of a sinusoidal function with midline along the horizontal axis when the amplitude is changed? How about when the other parameters are changed? Support your claim with evidence.
  - *The zeros of the function do not change when we change the amplitude because* $\sin\big(\omega(x-h)\big)=0$ *and* $A\sin\big(\omega(x-h)\big)=0$ *have the same solution sets. You can observe this by comparing graphs of functions* $f(x)=A\sin(\omega(x-h))$ *for different values of* A*. When you change the other parameters, the zeros change unless the changed parameter causes the graph of the new function to be a reflection of the original function, or the changed parameter causes the graph of the new function to be a horizontal translation of the original function that shifts the zeros onto themselves.*
- The final change is to consider the effect of $k=2$. This gives us the final graph, which is the graph of the function $f(x)=3\sin\left(4\left(x-\frac{\pi}{6}\right)\right)+2$. What happens to the graph of this function when $k=2$?
  - *The new graph is the previous graph translated vertically by* 2 *units.*



The blue curve is the graph of the function desired: $f(x)=3\sin\left(4\left(x-\frac{\pi}{6}\right)\right)+2$.

Exercise (10 minutes)

Students should now be in mixed groups with at least one "expert" on each parameter in each group. Assign one of the following exercises to each group. In addition to each student recording his or her work on the student handout, groups should record their graphs and responses in a format that enables presentation to the class after this exercise, either on personal white boards, chart paper, or a clean sheet of paper. Graph paper, if available, will help students to create their graphs more precisely.

After each group presents the assigned function, students can work any remaining problems or try them on their own if time permits. Otherwise, some of these problems can be either included as homework exercises or used on the second day should this lesson extend to two class periods.

Scaffolding:

For struggling students:

- Provide larger size graph paper, or consider providing each group with graph paper on which the graph of the sine function is provided. In advance, select appropriate scaling for groups based on their function.
- Create an anchor chart, and post it in a prominent location to provide visual support for how to determine each feature of the graph along with its formula.

Exercise

For each function, indicate the amplitude, frequency, period, phase shift, vertical translation, and equation of the midline. Graph the function together with a graph of the sine function $f(x) = \sin(x)$ on the same axes. Graph at least one full period of each function.

a. $g(x) = 3\sin(2x) - 1$

Since $A = 3$, the amplitude is 3.

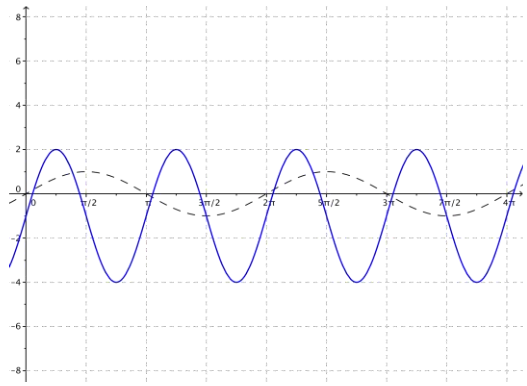
Since $\omega = 2$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{1}{\pi}$.

The period is $P = \frac{2\pi}{|\omega|} = \pi$.

Since $h = 0$, the phase shift is 0, and there is no horizontal translation.

The vertical translation is 1 unit downward since $k = -1$.

The midline has equation $y = -1$.



b. $g(x) = \frac{1}{2}\sin\left(\frac{1}{4}(x + \pi)\right)$

Since $A = \frac{1}{2}$, the amplitude is $\frac{1}{2}$.

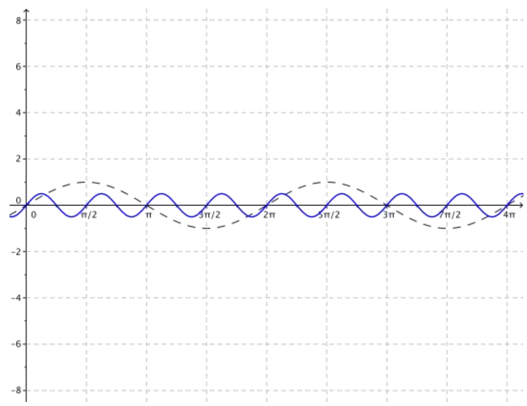
Since $\omega = \frac{1}{4}$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{1}{8\pi}$.

The period is $P = \frac{2\pi}{|\omega|} = 8\pi$.

Since $h = -\pi$, the phase shift is $-\pi$. Thus, the graph is translated horizontally π units to the left.

There is no vertical translation since $k = 0$.

The midline is the graph of $y = 0$.



c. $g(x) = 5\sin(-2x) + 2$

Since $A = 5$, the amplitude is 5.

Since $\omega = 2$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{1}{\pi}$.

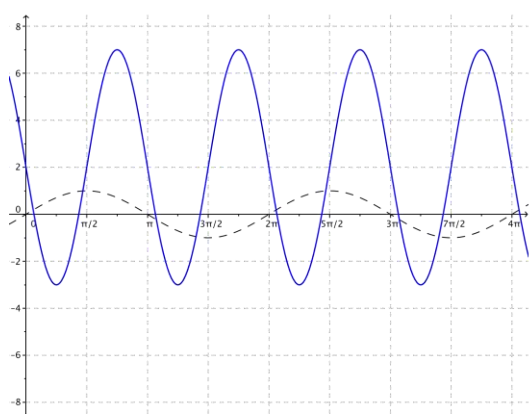
The period is $P = \frac{2\pi}{|\omega|} = \pi$.

Since the phase shift is $h = 0$, there is no horizontal translation.

Since $k = 2$, the vertical translation is 2 units upward.

The midline is the graph of $y = 2$.

The graph is reflected across the vertical axis since $\omega < 0$.



d. $g(x) = -2\sin\left(3\left(x + \frac{\pi}{6}\right)\right)$

Since $A = -2$, the amplitude is $|-2| = 2$.

Since $\omega = 3$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{3}{2\pi}$.

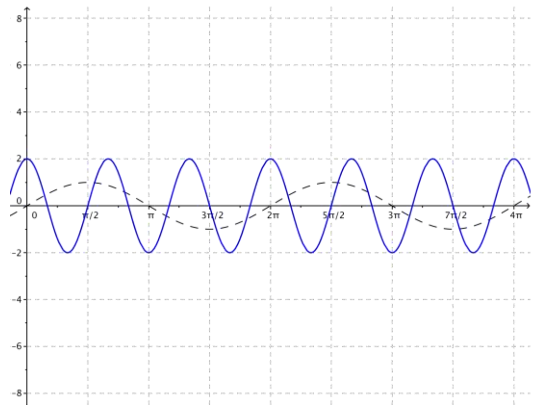
The period is $P = \frac{2\pi}{|\omega|} = \frac{2\pi}{3}$.

Since $h = \frac{\pi}{6}$, the phase shift is $\frac{\pi}{6}$. Thus, the graph is translated horizontally $\frac{\pi}{6}$ units to the right.

There is no vertical translation since $k = 0$.

The midline is the graph of $y = 0$.

The graph is reflected across the horizontal axis since $A < 0$.



e. $g(x) = 3\sin(x + \pi) + 3$

Since $A = 3$, the amplitude is 3.

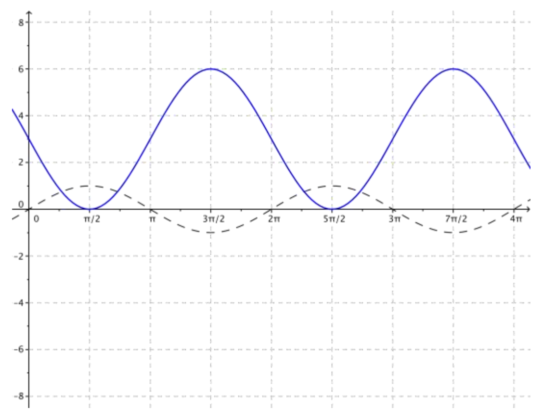
Since $\omega = 1$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{1}{2\pi}$.

The period is $P = \frac{2\pi}{|\omega|} = 2\pi$.

Since $h = -\pi$, the phase shift is $-\pi$. Thus, the graph is translated horizontally π units to the left.

Since $k = 3$, the graph is translated vertically 3 units upward.

The midline is the graph of $y = 3$.



f. $g(x) = -\frac{2}{3}\sin(4x) - 3$

Since $A = -\frac{2}{3}$, the amplitude is $\left|-\frac{2}{3}\right| = \frac{2}{3}$.

Since $\omega = 4$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{2}{\pi}$.

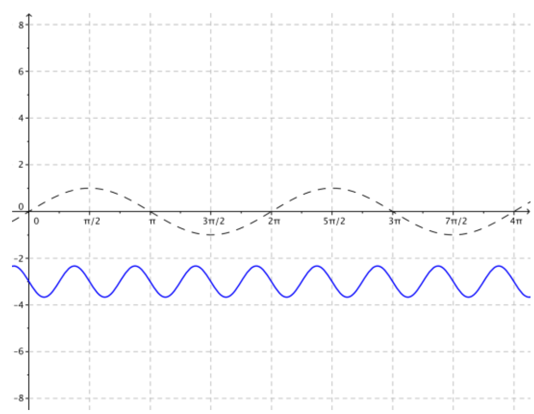
The period is $P = \frac{2\pi}{|\omega|} = \frac{\pi}{2}$.

Since $h = 0$, the phase shift is 0, and there is no horizontal translation.

Since $k = -3$, the vertical translation is 3 units downward.

The midline is the graph of $y = -3$.

Since $A < 0$, the graph is reflected about the horizontal axis.



g. $g(x) = \pi \sin\left(\frac{x}{2}\right) + \pi$

Since $A = \pi$, the amplitude is π.

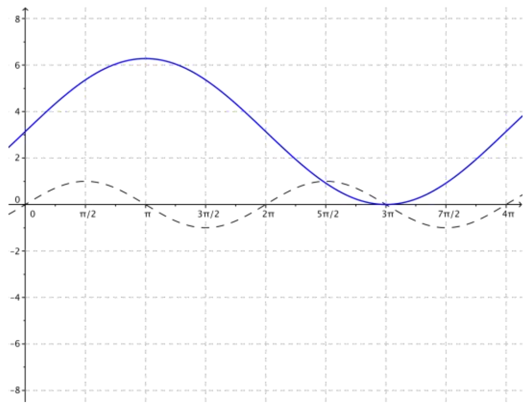
Since $\omega = \frac{1}{2}$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{1}{4\pi}$.

The period is $P = \frac{2\pi}{|\omega|} = 4\pi$.

Since $h = 0$, the phase shift is 0, and there is no horizontal translation.

Since $k = \pi$, the graph is translated vertically π units upward.

The midline is the graph of $y = \pi$.



h. $g(x) = 4 \sin\left(\frac{1}{2}(x - 5\pi)\right)$

Since $A = 4$, the amplitude is 4.

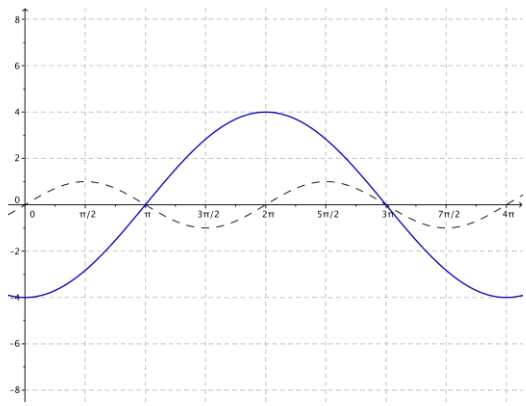
Since $\omega = \frac{1}{2}$, the frequency is $f = \frac{|\omega|}{2\pi} = \frac{1}{4\pi}$.

The period is $P = \frac{2\pi}{|\omega|} = 4\pi$.

Since $h = 5\pi$, the phase shift is 5π. Thus, the graph is translated horizontally 5π units to the right.

Since $k = 0$, there is no vertical translation.

The midline is the graph of $y = 0$.



Presentation (Optional—6 minutes)

Have one or two representatives from each group present the graphs from Exercise 1 to the class, explaining the value of the four parameters, A, ω, h, and k, from their function and how the graph of the sine function $f(x) = \sin(x)$ was affected by each parameter.

Closing (2 minutes)

Indicate that this reasoning extends to the graphs of the generalized cosine function $f(x) = A\cos\big(\omega(x-h)\big) + k$, except that the cosine graph is even, and the sine graph is odd. Thus, the graph of $y = \cos(-\omega x)$ and the graph of $y = \cos(\omega x)$ are the same graph. To sketch the graph of a cosine function, start with a graph of the cosine function $f(x) = \cos(x)$, and apply a series of transformations based on the values of the parameters to its graph to generate the graph of the given function.

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

This lesson investigated the effects of the parameters A, ω, h, and k on the graph of the function $f(x) = A\sin\big(\omega(x-h)\big) + k$.

- **The amplitude of the function is $|A|$; the vertical distance from a maximum point to the midline of the graph is $|A|$.**
- **The frequency of the function is $f = \frac{|\omega|}{2\pi}$, and the period is $P = \frac{2\pi}{|\omega|}$. The period is the vertical distance between two consecutive maximal points on the graph of the function.**
- **The phase shift is h. The value of h determines the horizontal translation of the graph from the graph of the sine function. If $h > 0$, the graph is translated h units to the right, and if $h < 0$, the graph is translated h units to the left.**
- **The graph of $y = k$ is the midline. The value of k determines the vertical translation of the graph compared to the graph of the sine function. If $k > 0$, then the graph shifts k units upward. If $k < 0$, then the graph shifts k units downward.**

These parameters affect the graph of $f(x) = A\cos\big(\omega(x-h)\big) + k$ similarly.

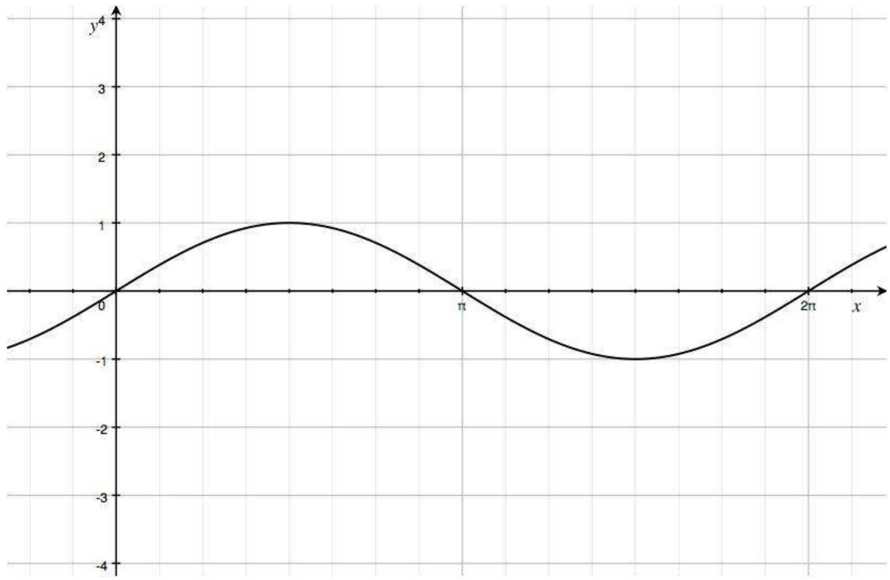
Exit Ticket (4 minutes)

Name____ Date____________________

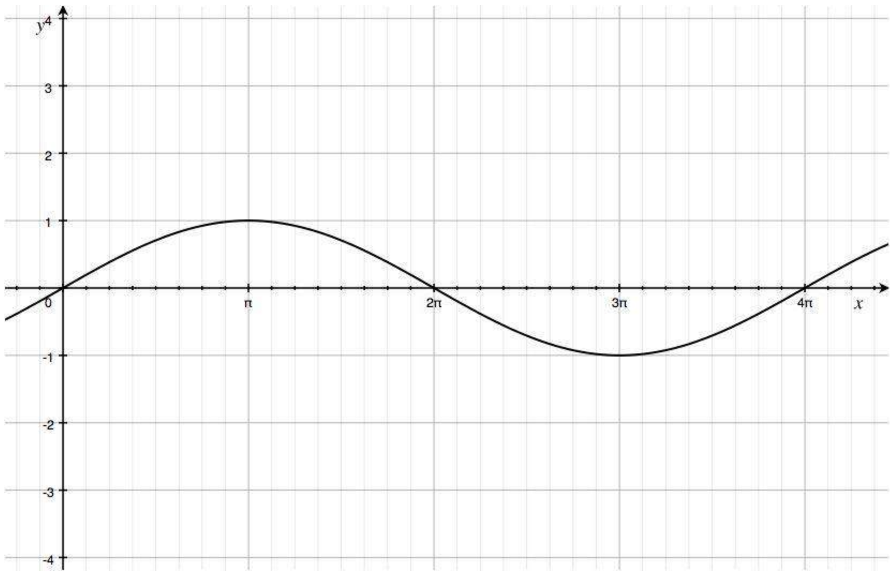
Lesson 11: Transforming the Graph of the Sine Function

Exit Ticket

1. Given the graph of $y = \sin(x)$ below, sketch the graph of the function $f(x) = \sin(4x)$ on the same set of axes. Explain the similarities and differences between the two graphs.



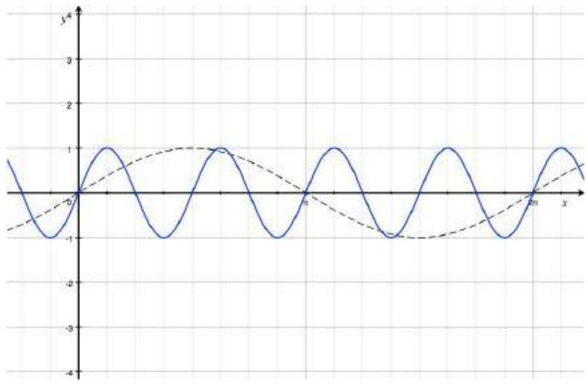
2. Given the graph of $y = \sin\left(\frac{x}{2}\right)$ below, sketch the graph of the function $g(x) = 3\sin\left(\frac{x}{2}\right)$ on the same set of axes. Explain the similarities and differences between the two graphs.



Exit Ticket Sample Solutions

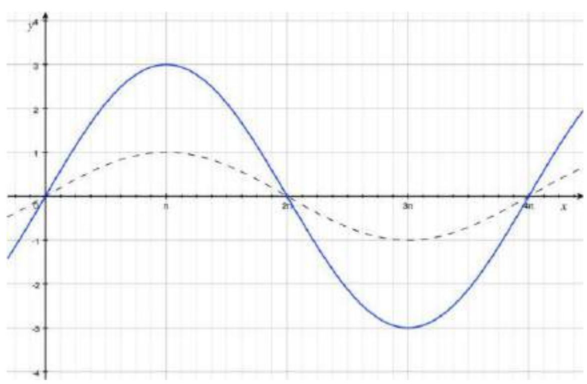
1. **Given the graph of $y = \sin(x)$ below, sketch the graph of the function $f(x) = \sin(4x)$ on the same set of axes. Explain the similarities and differences between the two graphs.**

   The curve shown in blue is the graph of $f(x) = \sin(4x)$, and the dashed curve is the graph of $y = \sin(x)$. The graph of f is a horizontal scaling of the graph of the sine function by a factor of $\frac{1}{4}$. The graph of $f(x) = \sin(4x)$ has a different period and frequency than the sine function, which changes the values of the x-intercepts, maximum and minimum points, and the increasing and decreasing intervals for this function. The amplitudes of the two graphs are the same, with $|A| = 1$.



2. **Given the graph of $y = \sin\left(\frac{x}{2}\right)$ below, sketch the graph of the function $g(x) = 3\sin\left(\frac{x}{2}\right)$ on the same set of axes. Explain the similarities and differences between the two graphs.**

   The curve shown in blue is the graph of $y = \sin\left(\frac{x}{2}\right)$, and the dashed curve is the graph of $g(x) = 3\sin\left(\frac{x}{2}\right)$. The graphs have different amplitudes. The graph of g is a vertical scaling of the graph of the sine function by a factor of 3. The y-coordinates of the maximum and minimum points are different for these two graphs. The two graphs have the same horizontal intercepts because the period of each function is 4π.

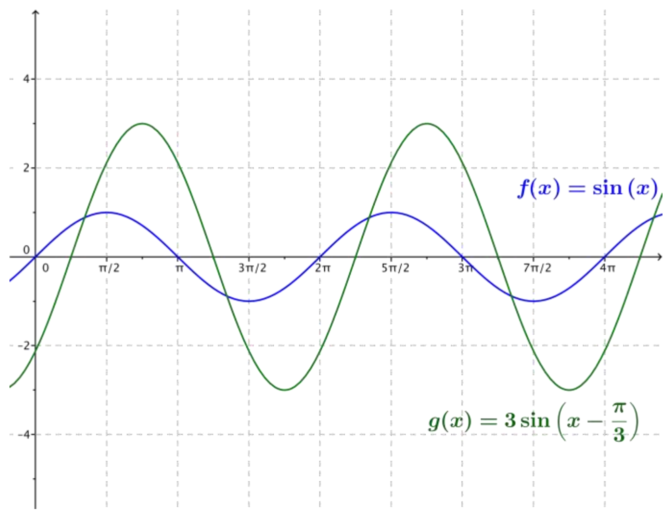


Problem Set Sample Solutions

1. For each function, indicate the amplitude, frequency, period, phase shift, horizontal and vertical translations, and equation of the midline. Graph the function together with a graph of the sine function $f(x) = \sin(x)$ on the same axes. Graph at least one full period of each function. No calculators are allowed.

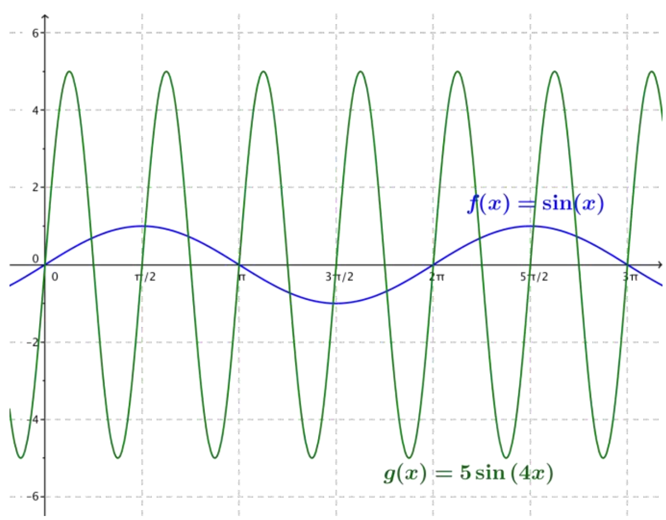
   a. $g(x) = 3\sin\left(x - \frac{\pi}{4}\right)$

   The amplitude is 3, the frequency is $\frac{1}{2\pi}$, the period is 2π, and the phase shift is $\frac{\pi}{4}$. The graph is translated horizontally $\frac{\pi}{4}$ units to the right; there is no vertical translation, and the equation of the midline is $y = 0$.



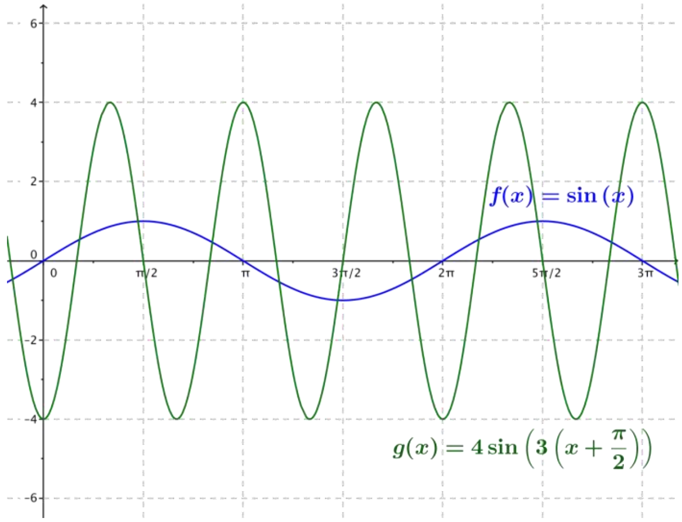
   b. $g(x) = 5\sin(4x)$

   The amplitude is 5, the frequency is $\frac{2}{\pi}$, the period is $\frac{\pi}{2}$, and the phase shift is 0. There are no horizontal or vertical translations, and the equation for the midline is $y = 0$. The graph is a vertical scaling of the graph of the sine function by a factor of 5 and a horizontal scaling of the sine function by a factor of $\frac{1}{4}$.



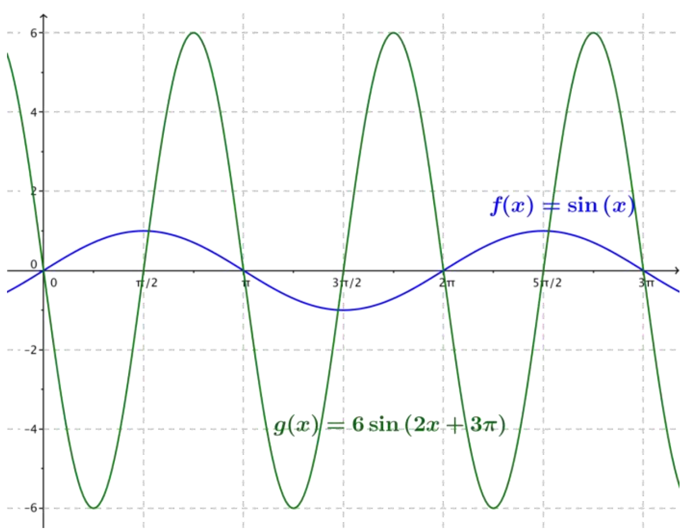
c. $g(x) = 4\sin\left(3\left(x + \frac{\pi}{2}\right)\right)$

The graph of f is the graph of the sine function scaled vertically by a factor of 4, horizontally by a factor of $\frac{1}{3}$, and translated $\frac{\pi}{2}$ to the left. The amplitude is 4, the frequency is $\frac{3}{2\pi}$, the period is $\frac{2\pi}{3}$, the phase shift is $-\frac{\pi}{2}$, and the equation for the midline is $y = 0$.



d. $g(x) = 6\sin(2x + 3\pi)$ (Hint: First, rewrite the function in the form $g(x) = A\sin(\omega(x - h))$.)

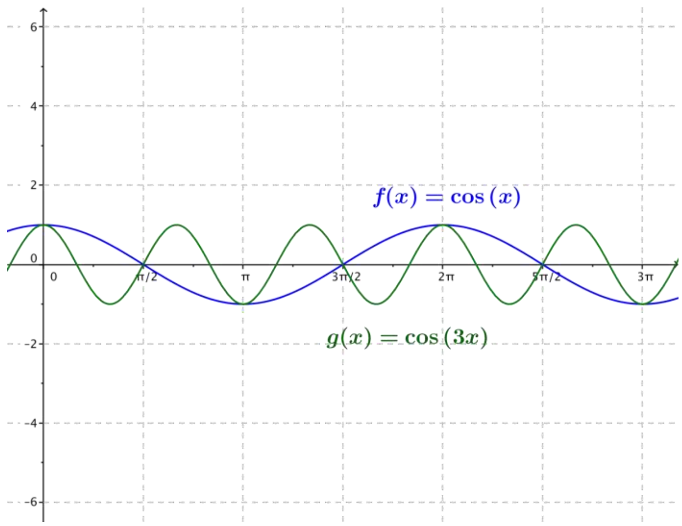
Rewriting the expression for g, we get $6\sin(2x + 3\pi) = 6\sin\left(2\left(x + \frac{3}{2}\pi\right)\right)$. The graph of g is the graph of the sine function scaled vertically by a factor of 6, horizontally by a factor of $\frac{1}{2}$, and translated horizontally $\frac{3}{2}\pi$ to the left. Thus, the amplitude is 6, the frequency is $\frac{1}{\pi}$, the period is π, the phase shift is $-\frac{3}{2}\pi$, and the equation for the midline is $y = 0$.



2. **For each function, indicate the amplitude, frequency, period, phase shift, horizontal and vertical translations, and equation of the midline. Graph the function together with a graph of the sine function $f(x) = \cos(x)$ on the same axes. Graph at least one full period of each function. No calculators are allowed.**

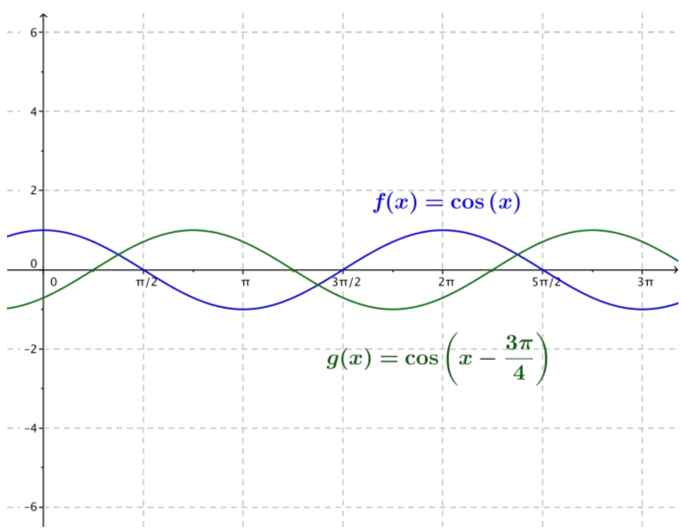
   a. $g(x) = \cos(3x)$

   The amplitude is 1, the frequency is $\frac{3}{2\pi}$, the period is $\frac{2}{3}\pi$, and the phase shift is 0. There are no horizontal or vertical translations, and the equation of the midline is $y = 0$.

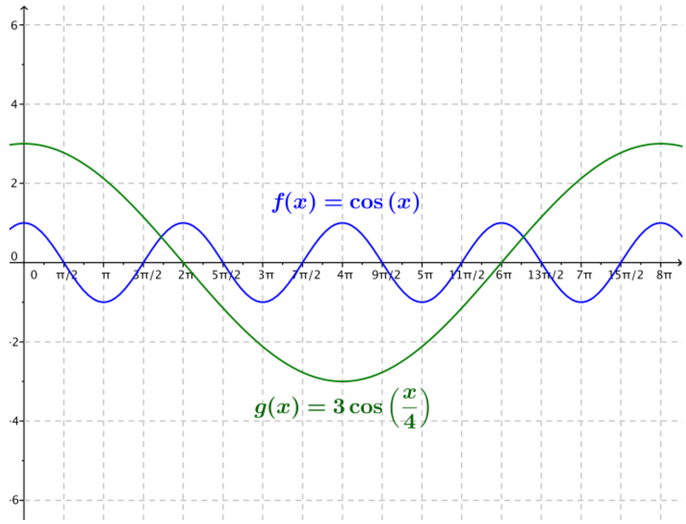


   b. $g(x) = \cos\left(x - \frac{3\pi}{4}\right)$

   The amplitude is 1, the frequency is $\frac{1}{2\pi}$, the period is 2π, and the phase shift is $\frac{3\pi}{4}$. The horizontal translation is $\frac{\pi}{3}$ units to the right, there is no vertical translation, and the equation of the midline is $y = 0$.



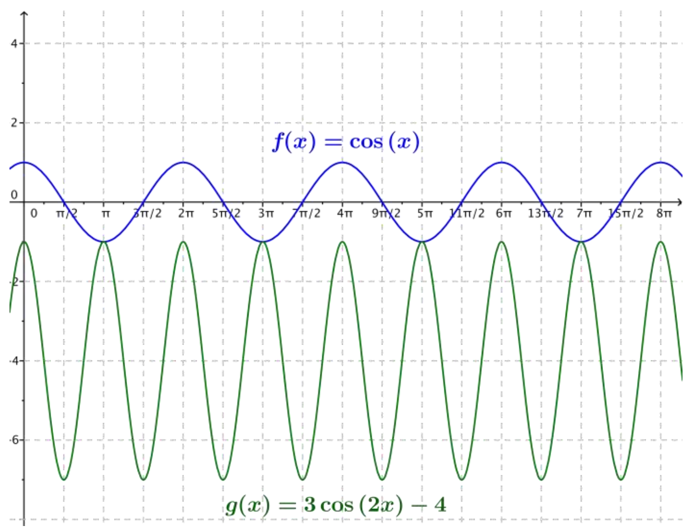
c. $g(x) = 3\cos\left(\frac{x}{4}\right)$



The amplitude is 3, the frequency is $\frac{1}{8\pi}$, the period is 8π, and the phase shift is 0. There are no horizontal or vertical translations, and the equation for the midline is $y = 0$. The graph of g is a vertical scaling of the graph of the cosine function by a factor of 3 and a horizontal scaling of the graph of the cosine function by a factor of 4.

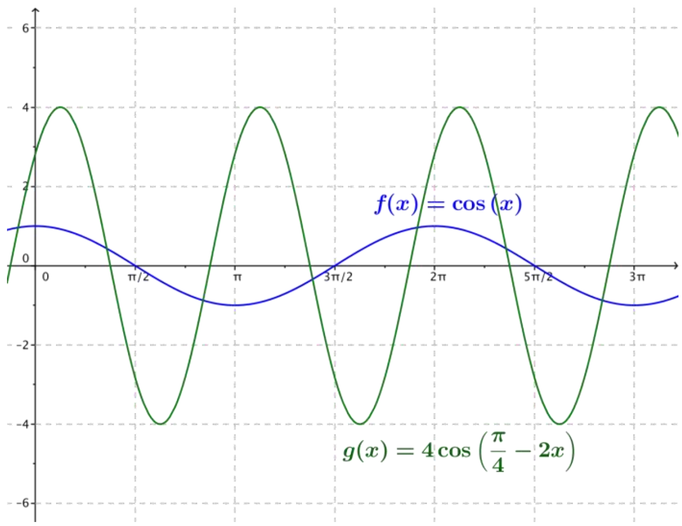
d. $g(x) = 3\cos(2x) - 4$

The amplitude is 3, the frequency is $\frac{1}{\pi}$, the period is π, and the phase shift is 0. There is no horizontal translation, the graph is translated vertically down 4 units, and the equation for the midline is $y = -4$.



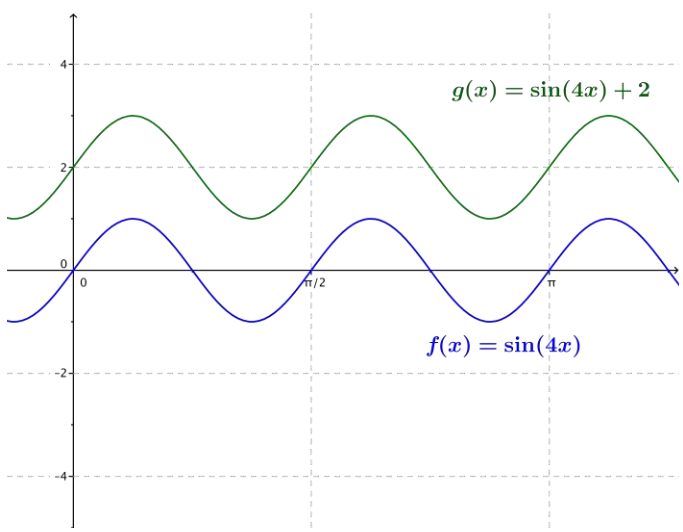
e. $g(x) = 4\cos\left(\frac{\pi}{4} - 2x\right)$ (Hint: First, rewrite the function in the form $g(x) = A\cos(\omega(x-h))$.)

Rewriting the expression, we see that $4\cos\left(\frac{\pi}{4} - 2x\right) = 4\cos\left(-2\left(x - \frac{\pi}{8}\right)\right) = 4\cos\left(2\left(x - \frac{\pi}{8}\right)\right)$. The graph of g is the graph of the cosine function scaled vertically by a factor of 4, horizontally by a factor of $\frac{1}{2}$, and then translated horizontally $\frac{\pi}{8}$ units to the right. This tells us that the amplitude is 4, the frequency is $\frac{1}{\pi}$, the period is π, the phase shift is $\frac{\pi}{8}$, and the equation for the midline is $y = 0$.

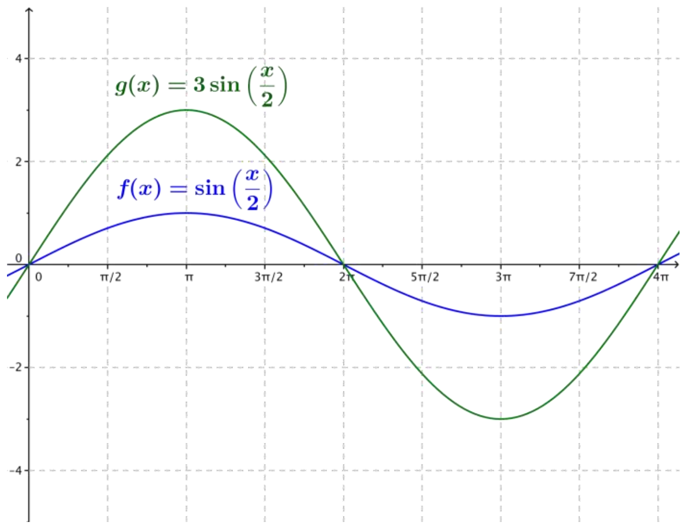


3. For each problem, sketch the graph of the pairs of indicated functions on the same set of axes without using a calculator or other graphing technology.

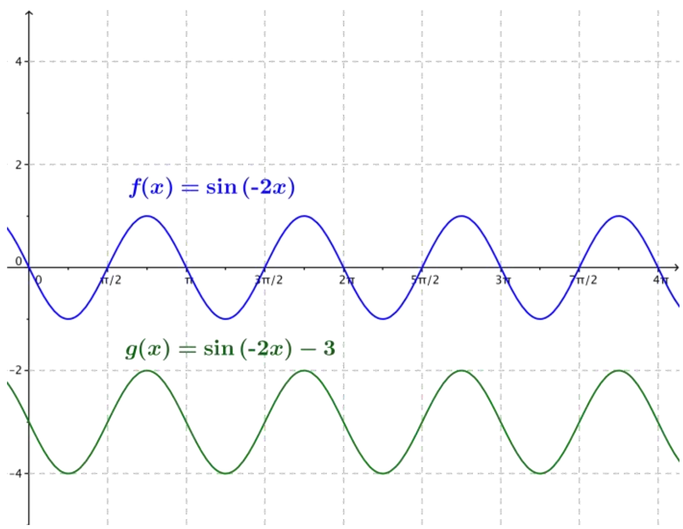
a. $f(x) = \sin(4x)$, $g(x) = \sin(4x) + 2$



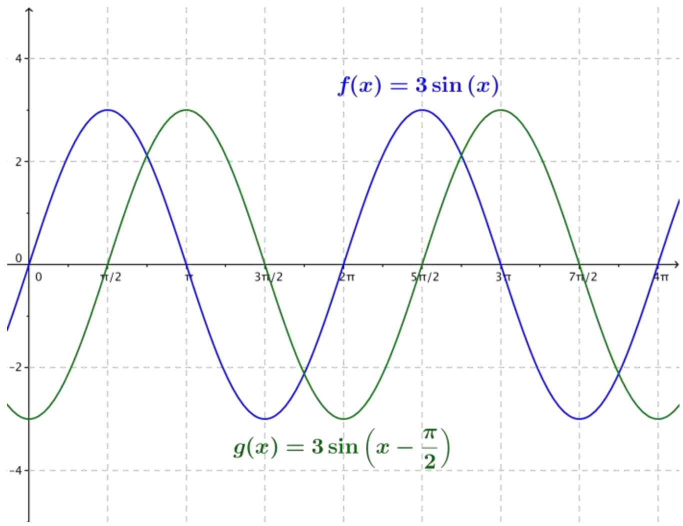
b. $f(x) = \sin\left(\frac{1}{2}x\right)$, $g(x) = 3\sin\left(\frac{1}{2}x\right)$



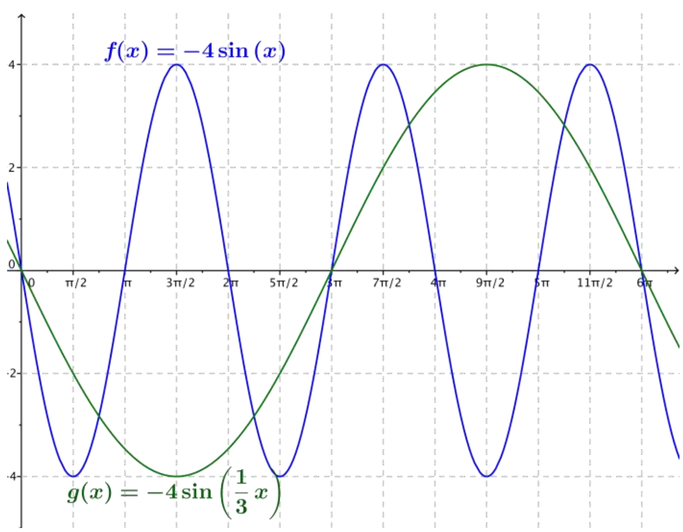
c. $f(x) = \sin(-2x)$, $g(x) = \sin(-2x) - 3$



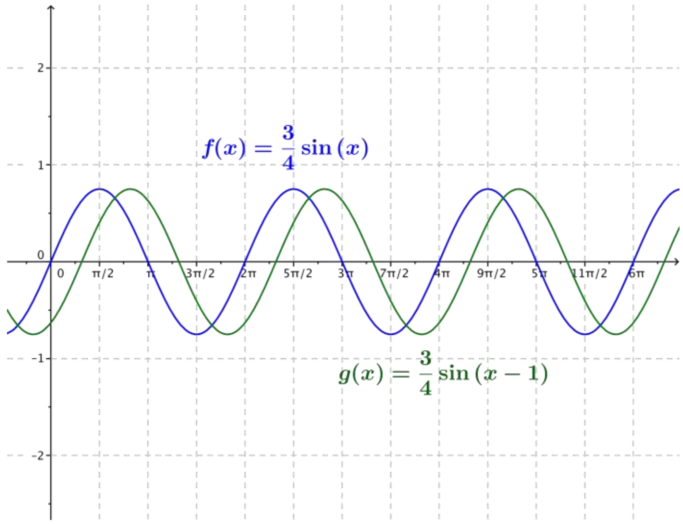
d. $f(x) = 3\sin(x)$, $g(x) = 3\sin\left(x - \frac{\pi}{2}\right)$



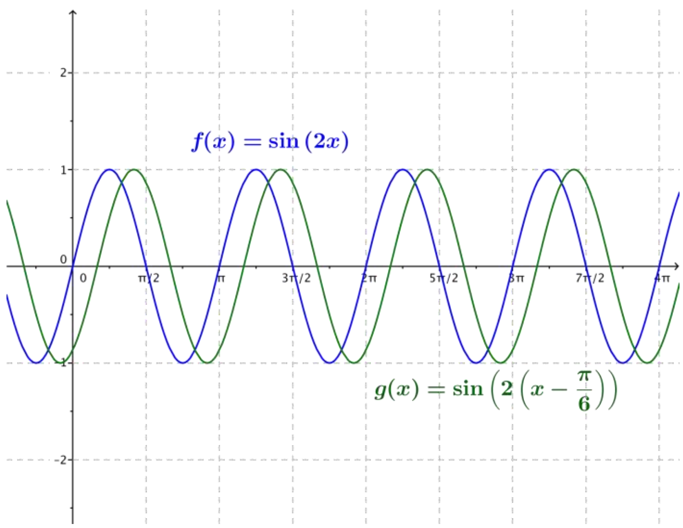
e. $f(x) = -4\sin(x)$, $g(x) = -4\sin\left(\frac{1}{3}x\right)$



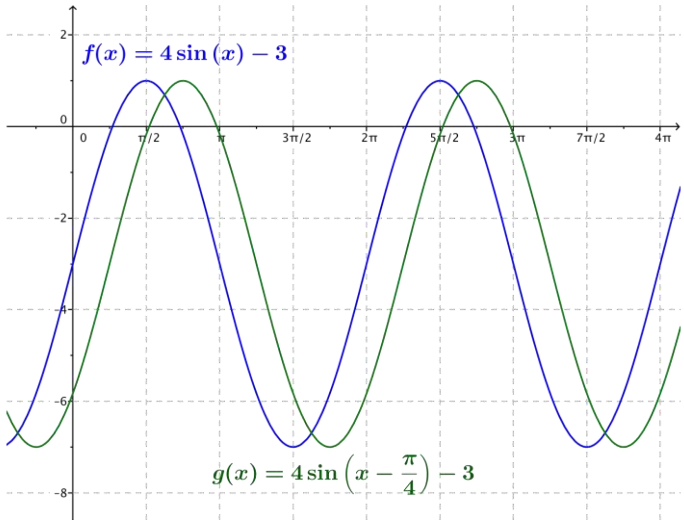
f. $f(x) = \frac{3}{4}\sin(x)$, $g(x) = \frac{3}{4}\sin(x - 1)$



g. $f(x) = \sin(2x)$, $g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$



h. $f(x) = 4\sin(x) - 3, g(x) = 4\sin\left(x - \frac{\pi}{4}\right) - 3$



Extension:

4. **Show that if the graphs of the functions $f(x) = A\sin(\omega(x - h_1)) + k$ and $g(x) = A\sin(\omega(x - h_2)) + k$ are the same, then h_1 and h_2 differ by an integer multiple of the period.**

Outline of proof: Since functions f and g have the same graph, the functions have the same period P. For all values of x, the fact that the sine graphs are the same gives

$$A\sin(\omega(x - h_1)) = A\sin(\omega(x - h_2))$$
$$\sin(\omega(x - h_1)) = \sin(\omega(x - h_2))$$
$$\omega(x - h_1) - \omega(x - h_2) = n2\pi$$
$$\frac{2\pi}{P}(x - h_1) - \frac{2\pi}{P}(x - h_2) = n2\pi$$
$$(x - h_1) - (x - h_2) = nP$$
$$h_1 - h_2 = nP.$$

5. **Show that if h_1 and h_2 differ by an integer multiple of the period, then the graphs of $f(x) = A\sin\big(\omega(x-h_1)\big)+k$ and $g(x) = A\sin\big(\omega(x-h_2)\big)+k$ are the same graph.**

Outline of proof:

Since h_1 and h_2 differ by an integer multiple of the period P, there is an integer n so that $h_2 - h_1 = nP$.

$$h_2 - h_1 = nP$$
$$(x-h_1)-(x-h_2) = nP$$
$$\frac{2\pi}{P}(x-h_1)-\frac{2\pi}{P}(x-h_2) = n2\pi$$
$$\omega(x-h_1)-\omega(x-h_2) = n2\pi$$
$$\omega(x-h_1) = \omega(x-h_2)+n2\pi$$
$$\sin\big(\omega(x-h_1)\big) = \sin(\omega(x-h_2)+n2\pi)$$

But, the sine function is periodic with period 2π, so $\sin(w(x-h_2)+n2\pi) = \sin\big(w(x-h_2)\big)$.

$$\sin\big(\omega(x-h_1)\big) = \sin\big(\omega(x-h_2)\big)$$
$$A\sin\big(\omega(x-h_1)\big) = A\sin\big(\omega(x-h_2)\big)$$
$$A\sin\big(\omega(x-h_1)\big)+k = A\sin\big(\omega(x-h_2)\big)+k$$
$$f(x) = g(x).$$

Since the functions f and g are the same, they have the same graph.

6. **Find the x-intercepts of the graph of the function $f(x) = A\sin\big(\omega(x-h)\big)$ in terms of the period P, where $\omega > 0$.**

The x-intercepts occur when $\sin\big(\omega(x-h)\big) = 0$. This happens when $\omega(x-h) = n\pi$ for integers n.

So, $x - h = \frac{n\pi}{\omega}$, and then $x = \frac{n\pi}{\omega} + h$.

Since $P = \frac{2\pi}{\omega}$, this becomes $x = \frac{nP}{2} + h$. Thus, the graph of $f(x) = A\sin(\omega(x-h))$ has x-intercepts at $x = \frac{n\pi}{\omega} + h$ for integer values of n.

Lesson 12: Ferris Wheels—Using Trigonometric Functions to Model Cyclical Behavior

Student Outcomes

- Students review how changing the parameters A, ω, h, and k in
$$f(x) = A\sin\big(\omega(x-h)\big) + k$$
affects the graph of a sinusoidal function.
- Students examine the example of the Ferris wheel, using height, distance from the ground, period, and so on, to write a function of the height of the passenger cars in terms of the sine function:
$$f(x) = A\sin\big(\omega(x-h)\big) + k.$$

Lesson Notes

In this lesson, students approach modeling with sinusoidal functions by extracting the equation from the graph instead of producing the graph from the equation, as done in the previous lesson. In Lesson 11, students studied the effect of changing the parameters A, ω, h, and k on the shape and position of the graph of a sinusoidal function of the form $f(x) = A\sin(\omega(x-h)) + k$. In this lesson, students return to the study of the height of a passenger car on a Ferris wheel, but they build up to the idea of the wheel's position as a function of *time* and not just as a function of the amount of rotation the wheel has undergone. In this way, students can create a more dynamic representation of the motion of the wheel. This change in perception is the result of composing functions—the sine is a function of the amount of rotation, which is in turn a function of the amount of time that has elapsed—but that should not be made explicit here. Because periodic situations are often based on time, thinking of sine and cosine as functions of time is the next natural step in student learning. Note that when the independent variable is time, frequency rates are expressed in terms of cycles per hour or cycles per minute. The addition of time as the independent variable allows students to model more complex, and hence more realistic, periodic phenomena (MP.4).

Parameterized functions are used in the Exploratory Challenge in a natural way; there is no need to teach to this idea directly. By this point, students should be familiar with the horizontal position of a car on the Ferris wheel being described by a cosine function and the vertical position of a car on the Ferris wheel being described by a sine function. It follows, then, that the position of the passenger car in the coordinate plane would be given by both a cosine function for the x-coordinate and a sine function for the y-coordinate. In particular, using a graphing calculator to graph the parametric equations for the position of a passenger car on the Ferris wheel presents a dynamic visual aid of a point tracing around the circle in the plane, which represents the car moving around the circle of the Ferris wheel. As this is the only lesson that requires the use of parameterized functions, the teacher might choose to model the calculator work in front of the class.

Classwork

Opening Exercise (3 minutes)

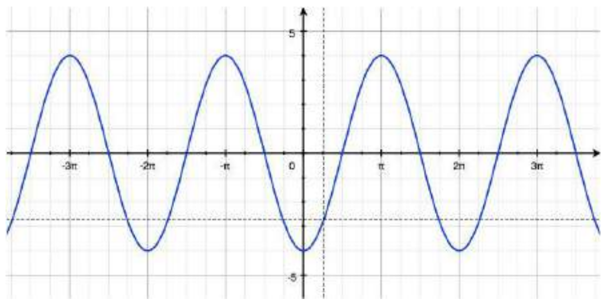
Show the graph, and pose the question to the class. Give them a minute to think quietly about the problem, and then ask for volunteers to answer the question.

MP.3

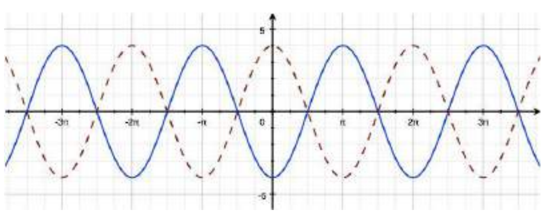
Opening Exercise

Ernesto thinks that this is the graph of $f(x) = 4\sin\left(x - \frac{\pi}{2}\right)$. Danielle thinks it is the graph of $f(x) = 4\cos(x)$.

Who is correct and why?



Ernesto is correct. For the function $y = A\sin(\omega(x-h)) + k$, Ernesto's solution would have $A = 4$, $\omega = 1$, $h = \frac{\pi}{2}$, and $k = 0$. The graph shown has an amplitude of 4 and a period of 2π; compared to the graph of the sine function, this graph is not translated vertically, making $k = 0$. This graph is a horizontal translation of the graph of the sine function to the right $\frac{\pi}{2}$ units. Meanwhile, for $y = A\cos(\omega(x-h)) + k$, Danielle's solution would have $A = 4$, $\omega = 1$, $h = 0$, and $k = 0$; however, the function $f(x) = 4\cos(x)$ has a maximum point at $(0, 4)$, and this graph has a minimum point when $x = 0$. To be correct, Danielle's function would have to be $f(x) = -4\cos(x)$. The dotted graph below is the graph of $f(x) = 4\cos(x)$.



In the discussion of the problem, be sure to remind students how the values of A, ω, h, and k affect the shape and position of the sine and cosine functions.

Scaffolding:

For students who need the graphing ideas from the previous lessons reinforced, use the following problems.

- Describe key features (periodicity, midline, amplitude, etc.) of the graph of $f(x) = 5\sin(x) + 4$.
- Graph these two functions on the same axes, and describe their key features.

  $f(x) = \cos(x)$

  $g(x) = \cos(2(x-1)) + 1$

Scaffolding:

Extensions for students who finish quickly:

- Write two trigonometric functions with the same zeros but different amplitudes.
- Write two trigonometric functions with different zeros but the same amplitude.
- Write two trigonometric functions with different zeros but the same amplitude and same period.
- Write two trigonometric functions with different periods but the same maximal and minimal values.

Exploratory Challenge/Exercises 1–5 (30 minutes)

This Exploratory Challenge revisits the Ferris wheel scenarios from prior lessons. The goal of this set of exercises is for students to work up to writing sinusoidal functions that give the height and co-height as functions of time, beginning with sketching graphs of the height and co-height functions of the Ferris wheel as previously done in Lessons 1 and 2 of this module. Have students split up into groups, and set them to work on the following exercises. In Exercise 4, students consider the motion of the Ferris wheel as a function of time, not of rotation. Be sure to clarify to students that the assumption is that the Ferris wheel rotates at a constant speed once the ride begins. In reality, the speed would increase from 0 ft/min to a fairly constant rate and then slowly decrease as the ride ends and the wheel comes to a stop.

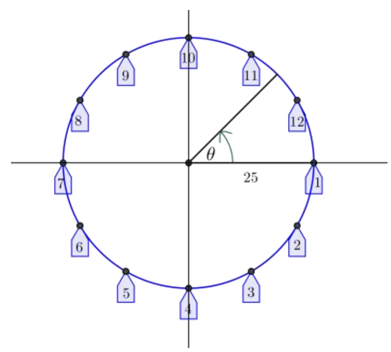
In these exercises, students encounter parameterized functions for the position of the Ferris wheel. They are using the capital letters X and Y to represent the functions for the horizontal and vertical components of the position of the wheel—what they have been calling the co-height and height functions—to distinguish from the variables x and y. In later courses, it is standard to use lowercase letters for these functions. Watching a graphing calculator draw the parameterized circle of the path of the wheel allows students to see the motion of the wheel as it completes its first turn.

If students do not frequently use graphing calculators, or if they are likely to have difficulty changing the calculator to parametric mode, then the first instance of graphing the parametric equations in Exercise 1(d) may require direction by the teacher. If graphing calculators are not available, then use online graphing software to graph the parametric equations; however, this software may not allow visualization of tracing the circle.

MP.4

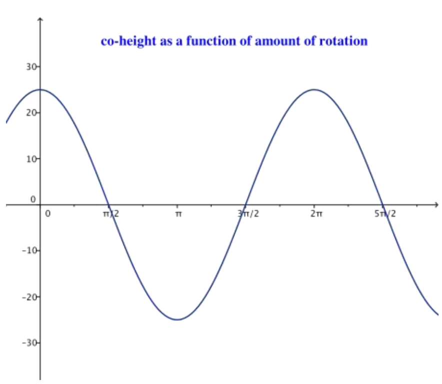
Exploratory Challenge/Exercises 1–5

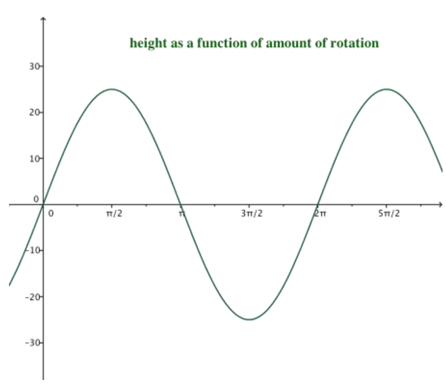
A carnival has a Ferris wheel that is 50 feet in diameter with 12 passenger cars. When viewed from the side where passengers board, the Ferris wheel rotates counterclockwise and makes two full turns each minute. Riders board the Ferris wheel from a platform that is 15 feet above the ground. We will use what we have learned about periodic functions to model the position of the passenger cars from different mathematical perspectives. We will use the points on the circle in the diagram on the right to represent the position of the cars on the wheel.



1. For this exercise, we will consider the height of a passenger car to be the vertical displacement from the horizontal line through the center of the wheel and the co-height of a passenger car to be the horizontal displacement from the vertical line through the center of the wheel.

a. Let $\theta = 0$ represent the position of car 1 in the diagram above. Sketch the graphs of the co-height and the height of car 1 as functions of θ, the number of radians through which the car has rotated.





MP.4

b. What is the amplitude, $|A|$, of the height and co-height functions for this Ferris wheel?

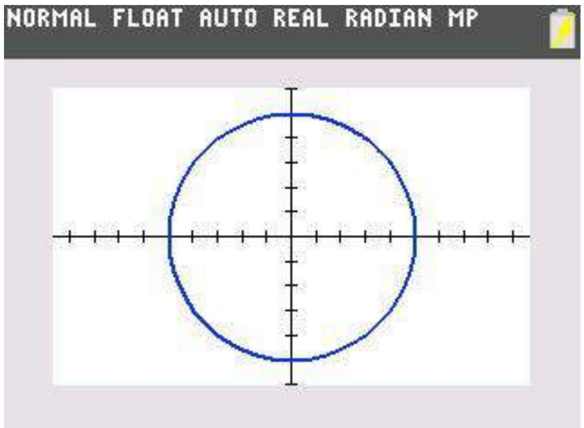
$|A| = 25$

c. Let $X(\theta)$ represent the co-height function after rotation by θ radians, and let $Y(\theta)$ represent the height function after rotation by θ radians. Write functions X for the co-height and Y for the height in terms of θ.

$X(\theta) = 25\cos(\theta)$
$Y(\theta) = 25\sin(\theta)$

MP.5

d. Graph the functions X and Y from part (c) on a graphing calculator set to parametric mode. Use a viewing window $[-48, 48] \times [-30, 30]$. Sketch the graph below.



e. Why did we choose the symbols X and Y to represent the co-height and height functions?

The co-height function measures the horizontal position of the passenger car that we typically name x. The height function measures the vertical position of the passenger car that we typically name y. It makes sense to use X as the name of the function that describes horizontal movement and Y as the name of the function that describes vertical movement.

MP.2

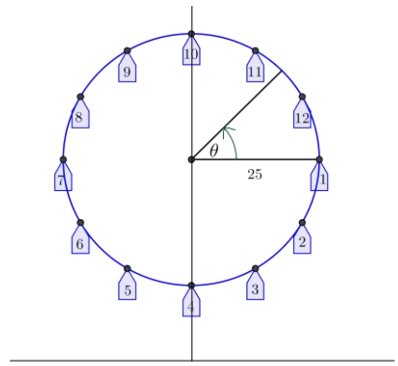
f. Evaluate $X(0)$ and $Y(0)$, and explain their meaning in the context of the Ferris wheel.

Using the equations from part (c), $X(0) = 25\cos(0) = 25$, and $Y(0) = 25\sin(0) = 0$. Thus, when the wheel has not yet rotated (that is, rotated through 0 radians), the co-height of the car is 25, and the height is 0. This means that the car is at the 3 o'clock position.

g. Evaluate $X\left(\frac{\pi}{2}\right)$ and $Y\left(\frac{\pi}{2}\right)$, and explain their meaning in the context of the Ferris wheel.

Using the equations from part (c), $X\left(\frac{\pi}{2}\right) = 25\cos\left(\frac{\pi}{2}\right) = 0$, and $Y\left(\frac{\pi}{2}\right) = 25\sin\left(\frac{\pi}{2}\right) = 25$. Thus, when the wheel has rotated through $\frac{\pi}{2}$ radians (that is, rotated through a quarter turn), the co-height of the car is 0, and the height is 25. This means that the car is at the top of the wheel.

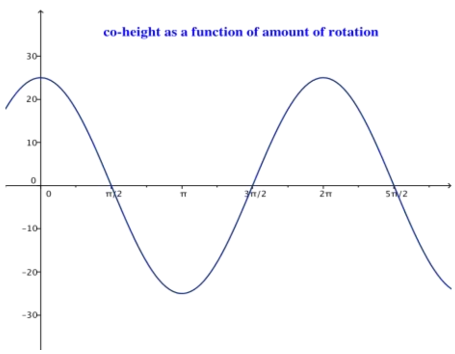
2. The model we created in Exercise 1 measures the height of car 1 above the horizontal line through the center of the wheel. We now want to alter this model so that it measures the height of car 1 above the ground.

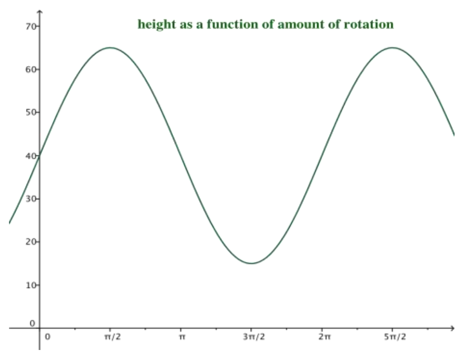


a. If we measure the height of car 1 above the ground instead of above the horizontal line through the center of the wheel, how will the functions X and Y need to change?

The co-height function X will not change, but the height function Y will need to have a constant added to it since car 1 starts 40 feet above the ground.

b. Let $\theta = 0$ represent the position of car 1 in the diagram to the right. Sketch the graphs of the co-height and the height of car 1 as functions of the number of radians through which the car has rotated, θ.





c. How are the graphs from part (b) related to the graphs from Exercise 1(a)?

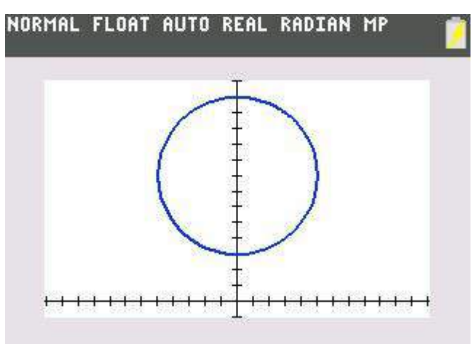
The co-height graphs are the same. The height graph from part (b) is the height graph from Exercise 1(a) translated 40 units upward.

d. From this perspective, find the equations for the functions X and Y that model the position of car 1 with respect to the number of radians the car has rotated.

$$X(\theta) = 25\cos(\theta)$$
$$Y(\theta) = 25\sin(\theta) + 40$$

Change the viewing window on your calculator to $[-60, 60] \times [-5, 70]$, and graph the functions X and Y together. Sketch the graph.

MP.2



MP.2

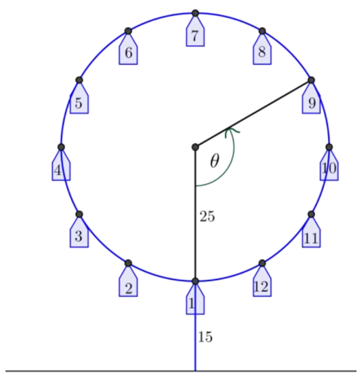
e. **Evaluate $X(0)$ and $Y(0)$, and explain their meaning in the context of the Ferris wheel.**

Using the equations from part (c), $X(0) = 25\cos(0) = 25$, and $Y(0) = 25\sin(0) + 40 = 40$. Thus, when the wheel has not yet rotated (that is, rotated through 0 radians), the co-height of the car is 25, and the height is 40. This means that the car is at the 3 o'clock position.

f. **Evaluate $X\left(\frac{\pi}{2}\right)$ and $Y\left(\frac{\pi}{2}\right)$, and explain their meaning in the context of the Ferris wheel.**

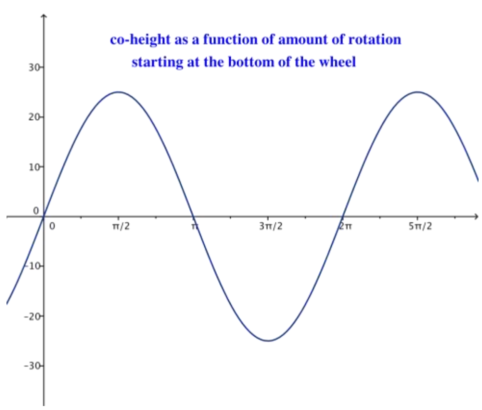
Using the equations from part (c), $X\left(\frac{\pi}{2}\right) = 25\cos\left(\frac{\pi}{2}\right) = 0$, and $Y\left(\frac{\pi}{2}\right) = 25\sin\left(\frac{\pi}{2}\right) + 40 = 65$. Thus, when the wheel has rotated through $\frac{\pi}{2}$ radians (that is, rotated through a quarter turn), the co-height of the car is 0, and the height is 65. This means that the car is at the top of the wheel.

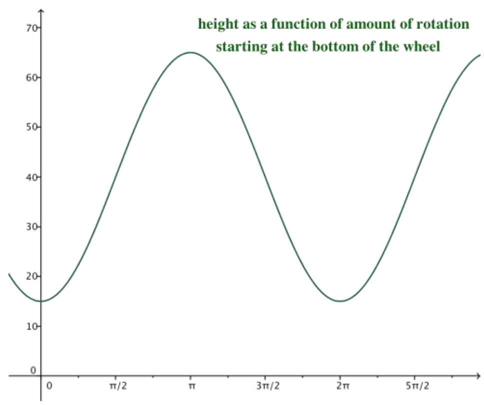
3. **In reality, no one boards a Ferris wheel halfway up; passengers board at the bottom of the wheel. To truly model the motion of a Ferris wheel, we need to start with passengers on the bottom of the wheel. Refer to the diagram below.**



Scaffolding:
Students can make a table of values to help sketch the co-height and height graphs.

a. **Let $\theta = 0$ represent the position of car 1 at the bottom of the wheel in the diagram above. Sketch the graphs of the height and the co-height of car 1 as functions of θ, the number of radians through which the car has rotated from the position at the bottom of the wheel.**





b. **How are the graphs from part (a) related to the graphs from Exercise 2(b)?**

The graphs from part (a) are the graphs from Exercise 2(b) translated horizontally $\frac{\pi}{2}$ units to the right.

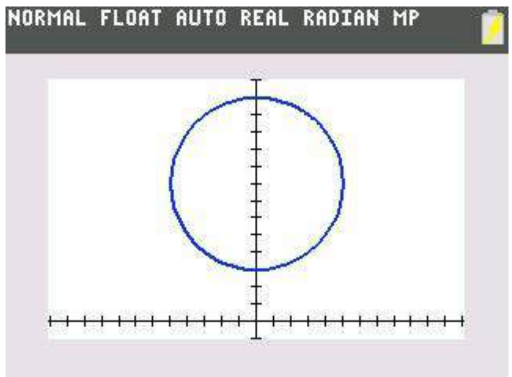
c. **From this perspective, find the equations for the functions X and Y that model the position of car 1 with respect to the number of radians the car has rotated.**

$$X(\theta) = 25\cos\left(\theta - \frac{\pi}{2}\right)$$

$$Y(\theta) = 25\sin\left(\theta - \frac{\pi}{2}\right) + 40$$

d. **Graph the functions X and Y from part (c) together on the graphing calculator. Sketch the graph. How is this graph different from the one from Exercise 2(e)?**

Although the finished graph looks exactly like the graph from Exercise 2(e), this one starts drawing from the bottom of the circle, as opposed to drawing from the 3 o'clock position.



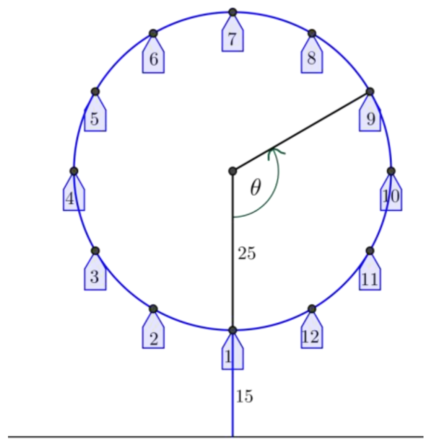
e. **Evaluate $X(0)$ and $Y(0)$, and explain their meaning in the context of the Ferris wheel.**

Using the equations from part (c), $X(0) = 25\cos\left(0 - \frac{\pi}{2}\right) = 0$, and $Y(0) = 25\sin\left(0 - \frac{\pi}{2}\right) + 40 = 15$. Thus, when the wheel has not yet rotated (that is, rotated through 0 radians), the co-height of the car is 0, and the height is 15. This means that the car is at the bottom of the wheel.

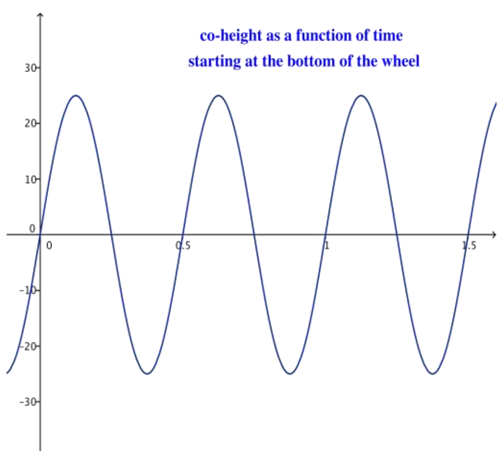
f. **Evaluate $X\left(\frac{\pi}{2}\right)$ and $Y\left(\frac{\pi}{2}\right)$, and explain their meaning in the context of the Ferris wheel.**

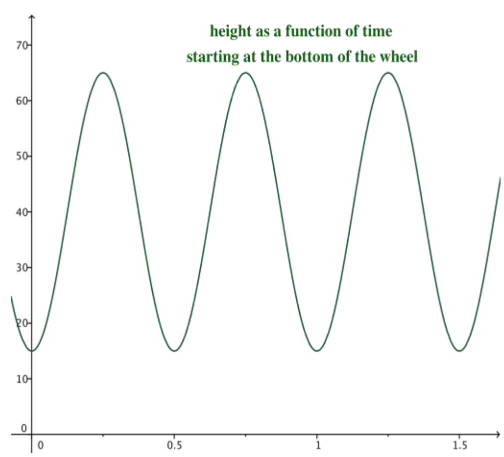
Using the equations from part (c), $X\left(\frac{\pi}{2}\right) = 25\cos\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 25$, and $Y\left(\frac{\pi}{2}\right) = 25\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + 40 = 40$. Thus, when the wheel has rotated through $\frac{\pi}{2}$ radians (that is, rotated through a quarter turn), the co-height of the car is 25, and the height is 40. This means that the car is at the 3 o'clock position.

4. **Finally, it is not very useful to track the position of a Ferris wheel as a function of how much it has rotated. It would make more sense to keep track of the Ferris wheel as a function of time. Recall that the Ferris wheel completes two full turns per minute.**



a. **Let $\theta = 0$ represent the position of car 1 at the bottom of the wheel. Sketch the graphs of the co-height and the height of car 1 as functions of time.**





b. **The co-height and height functions from part (a) can be written in the form $X(t) = A\cos(\omega(t-h)) + k$ and $Y(t) = A\sin(\omega(t-h)) + k$. From the graphs in part (a), identify the values of A, ω, h, and k for each function X and Y.**

For the co-height function: $A = 25$, $\omega = 4\pi$, $h = \frac{1}{8}$, *and* $k = 0$.

For the height function: $A = 25$, $\omega = 4\pi$, $h = \frac{1}{8}$, *and* $k = 40$.

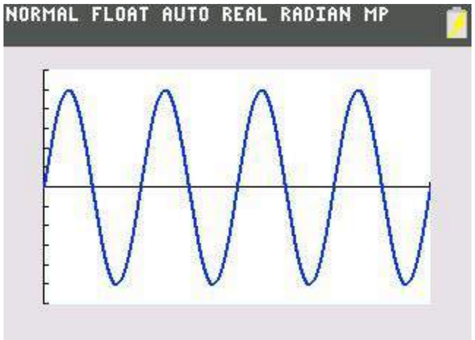
c. **Write the equations $X(t)$ and $Y(t)$ using the values you identified in part (b).**

$$X(t) = 25\cos\left(4\pi\left(t - \frac{1}{8}\right)\right)$$

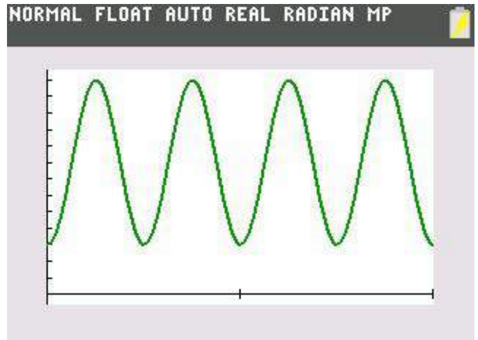
$$Y(t) = 25\sin\left(4\pi\left(t - \frac{1}{8}\right)\right) + 40$$

d. **In function mode, graph your functions from part (c) on a graphing calculator for $0 < t < 2$, and compare against your sketches in part (a) to confirm your equations.**

Co-height function, X



Height function, Y



e. **Explain the meaning of the parameters in your equation for X in terms of the Ferris wheel scenario.**

For the co-height function, we have $A = 25$, $\omega = 4\pi$, $h = \frac{1}{8}$, and $k = 0$.

- *The amplitude is $A = 25$, which means that the difference between the largest and smallest values of the co-height is 50 feet. This is the diameter of the wheel.*
- *The period is $P = \frac{2\pi}{|\omega|} = \frac{1}{2}$, and the frequency is $f = \frac{1}{P} = 2$. This means that the wheel rotates twice each minute.*
- *The phase shift is $h = \frac{1}{8}$. Each quarter revolution takes $\frac{1}{8}$ of a minute, which is 12.5 seconds. Without this shift, our equations would start car 1 in the 3 o'clock position. We want to start it from the bottom of the wheel.*
- *The equation of the midline is $y = 0$, which means that the co-height varies between -25 and 25 feet.*

f. **Explain the meaning of the parameters in your equation for Y in terms of the Ferris wheel scenario.**

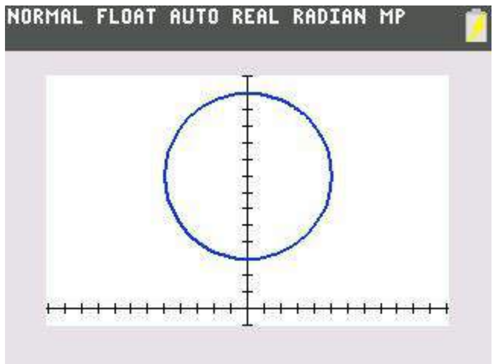
For the height function, we have $A = 25$, $\omega = 4\pi$, $h = \frac{1}{8}$, and $k = 40$.

- *The amplitude is $A = 25$, which means that the difference between the largest and smallest values of the co-height is 50 feet. This is the diameter of the wheel.*
- *The period is $P = \frac{2\pi}{|\omega|} = \frac{1}{2}$, and the frequency is $f = \frac{1}{P} = 2$. This means that the wheel rotates twice each minute.*
- *The phase shift is $h = \frac{1}{8}$. Each quarter revolution takes $\frac{1}{8}$ of a minute, which is 12.5 seconds. Without this shift, our equations would start car 1 in the 3 o'clock position. We want to start it from the bottom of the wheel.*
- *The equation of the midline is $y = 40$, which means that the height varies between 15 and 65 feet.*

5. In parametric mode, graph the functions X and Y from Exercise 3(c) on a graphing calculator for $0 \le t \le \frac{1}{2}$.

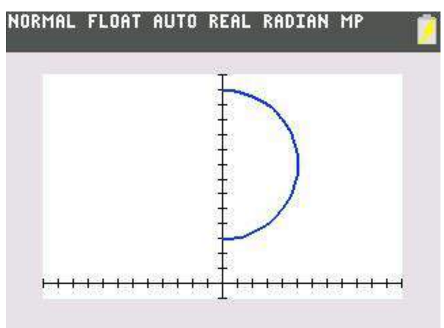
a. Sketch the graph. How is this graph different from the graph in Exercise 3(d)?

Although the finished graphs look identical, the difference is that this graph has a different scale for the parameter t. Instead of needing $0 < \theta < 2\pi$ to complete the circle, we only need to specify $0 < t < \frac{1}{2}$ since the wheel makes one complete turn in $\frac{1}{2}$ minute.



b. What would the graph look like if you graphed the functions X and Y from Exercise 3(c) for $0 \le t \le \frac{1}{4}$? Why?

If we graphed X and Y for $0 \le t \le \frac{1}{4}$, then the graph would only show the left half of the circle. This is because it takes $\frac{1}{4}$ minute for the car to travel from the bottom of the circle to the top.



c. Evaluate $X(0)$ and $Y(0)$, and explain their meaning in the context of the Ferris wheel.

Using the equations from part (c), $X(0) = 25\cos\left(0 - \frac{\pi}{2}\right) = 0$, and $Y(0) = 25\sin\left(0 - \frac{\pi}{2}\right) + 40 = 15$. Thus, when the wheel has not yet rotated (that is, rotated for 0 minutes), the co-height of the car is 0, and the height is 15. This means that the car is at the bottom of the wheel.

d. Evaluate $X\left(\frac{1}{8}\right)$ and $Y\left(\frac{1}{8}\right)$, and explain their meaning in the context of the Ferris wheel.

Using the equations from part (c), $X\left(\frac{1}{8}\right) = 25\cos\left(4\pi\left(\frac{1}{8}\right) - \frac{\pi}{2}\right) = 25$, and $Y\left(\frac{1}{8}\right) = 25\sin\left(4\pi\left(\frac{1}{8}\right) - \frac{\pi}{2}\right) + 40 = 40$. Thus, when the wheel has rotated for $\frac{1}{8}$ of a minute (that is, rotated through a quarter turn since the wheel rotates once every $\frac{1}{2}$ minute), the co-height of the car is 25, and the height is 40. This means that the car is at the 3 o'clock position.

Exercises 6–9 (5 minutes)

Omit Exercises 6–9 if time is short. Groups who have quickly completed the first set of exercises could begin these exercises, or they may be assigned as additional homework.

Exercises 6–9

6. You are in car 1, and you see your best friend in car 3. How much higher than your friend are you when you reach the top?

When you are at the top, your height is 65 feet, and you have rotated through $180°$, which is π radians. At that time, your friend in car 3 has rotated through $60°$ less than you have, so car 3 has rotated through $120°$, which is $\frac{2\pi}{3}$ radians. Using the height as a function of rotation from Exercise 3, we know that the height of car 3 is

$$\begin{aligned} Y\left(\frac{2\pi}{3}\right) &= 25\sin\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)+40 \\ &= 25\left(\sin\left(\frac{\pi}{6}\right)\right)+40 \\ &= 52.5. \end{aligned}$$

You are 65 feet high, and your friend it 52.5 feet high; therefore, you are 12.5 feet higher than your friend in car 3.

7. Find an equation of the function H that represents the difference in height between you in car 1 and your friend in car 3 as the wheel rotates through θ radians, beginning with $\theta = 0$ at the bottom of the wheel.

The height of car 1 after rotation through θ radians is given by $Y(\theta)$. Since car 3 is $\frac{\pi}{3}$ radians behind car 1, the height of car 3 after rotation through θ radians is given by $Y\left(\theta-\frac{\pi}{3}\right)$. Then, the height difference is given by the equation:

$$\begin{aligned} H(\theta) &= Y(\theta) - Y\left(\theta-\frac{\pi}{3}\right) \\ &= \left(25\sin\left(\theta-\frac{\pi}{2}\right)+40\right)-\left(25\sin\left(\theta-\frac{\pi}{3}-\frac{\pi}{2}\right)+40\right) \\ &= 25\left(\sin\left(\theta-\frac{\pi}{2}\right)-\sin\left(\theta-\frac{5\pi}{6}\right)\right). \end{aligned}$$

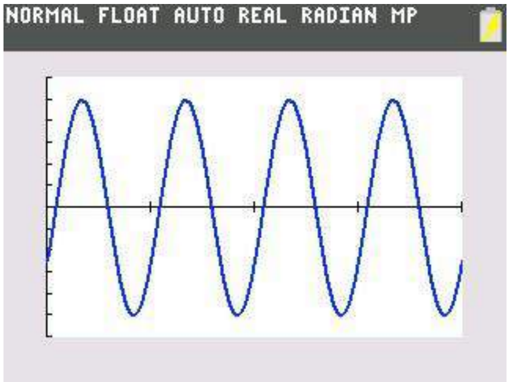
8. Find an equation of the function that represents the difference in height between car 1 and car 3 with respect to time, t, in minutes. Let $t = 0$ minutes correspond to a time when car 1 is located at the bottom of the Ferris wheel. Assume the wheel is moving at a constant speed starting at $t = 0$.

Car 3 is $\frac{1}{6}$ of a turn behind car 1. Since it takes $\frac{1}{2}$ minute to make one turn, car 3 is $\frac{1}{6}\cdot\frac{1}{2}=\frac{1}{12}$ of a minute behind car 1. Then, at time t, the position of car 1 is $Y(t)$, and the position of car 3 is $Y\left(t-\frac{1}{12}\right)$.

$$\begin{aligned} H(t) &= Y(t) - Y\left(t-\frac{1}{12}\right) \\ &= \left[25\sin\left(4\pi\left(t-\frac{1}{8}\right)\right)+40\right]-\left[25\sin\left(4\pi\left(t-\frac{1}{12}-\frac{1}{8}\right)\right)+40\right] \\ &= 25\left(\sin\left(4\pi\left(t-\frac{1}{8}\right)\right)-\sin\left(4\pi\left(t-\frac{5}{24}\right)\right)\right). \end{aligned}$$

9. Use a calculator to graph the function H in Exercise 8 for $0 \leq t \leq 2$. What type of function does this appear to be? Does that make sense?

The function H appears to be a periodic sinusoidal function. It makes sense that it would be periodic because the distance between the two cars would be the same each time the wheel makes a complete turn. That is, every $\frac{1}{2}$ minute, the distance between the cars repeats.



The reality is that no one would actually have a need to calculate this distance, especially when enjoying a ride on a Ferris wheel. However, the point of Exercises 6–9 is that the distance between these cars at any time t can be modeled by subtracting two sinusoidal functions. The difference between two sinusoids *does* have many interesting applications when studying more complex waveforms in physics, such as light, radio, acoustic, and surface water waves. The sum and difference formulas that students study in Precalculus and Advanced Topics explain why the difference is also a sinusoidal function.

Finally, ask students why it is useful to have models such as this one.

- Why would anyone want to model the height of a passenger car on a Ferris wheel? More generally, what might be the value of studying models of circular motion?
  - *Perhaps knowing the precise height as a function of time might be useful for aesthetic reasons or safety reasons that have to do with design or engineering features. In general, the motion of any object traveling in a circular path can be modeled by a sinusoidal function, including many real-world situations, such as the motion of a pendulum or an engine's piston-crankshaft.*

Closing (3 minutes)

Display the height function derived by students for car 1 of the Ferris wheel in Exercise 4.

$$H(t) = 25\sin\left(4\pi\left(t - \frac{1}{8}\right)\right) + 40.$$

Then, lead students through this closing discussion.

- How would this formula change for a Ferris wheel with a different diameter?
  - *The* 25 *would change to the radius of the Ferris wheel.*
- How would this formula change for a Ferris wheel at a different height off the ground?
  - *The* 40 *would change to the measurement from the ground to the central axis of the Ferris wheel.*
- How would this formula change for a Ferris wheel that had a different rate of revolution?
  - *The number of revolutions per minute would change, so the period and frequency would change.*
- How would this formula change if we modeled the height of a passenger car above the ground from a different starting position on the wheel?
  - *The height at the time corresponding to* $t = 0$ *would change, so changing the phase shift could horizontally translate the function to have the correct height correspond to the starting time.*

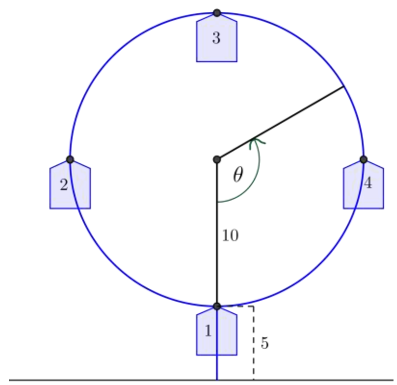
Exit Ticket (4 minutes)

Name ______________________________ Date ________________

Lesson 12: Ferris Wheels—Using Trigonometric Functions to Model Cyclical Behavior

Exit Ticket

The Ferris Wheel Again



In an amusement park, there is a small Ferris wheel, called a kiddie wheel, for toddlers. The points on the circle in the diagram to the right represent the position of the cars on the wheel. The kiddie wheel has four cars, makes one revolution every minute, and has a diameter of 20 feet. The distance from the ground to a car at the lowest point is 5 feet. Assume $t = 0$ corresponds to a time when car 1 is closest to the ground.

1. Sketch the height function for car 1 with respect to time as the Ferris wheel rotates for two minutes.

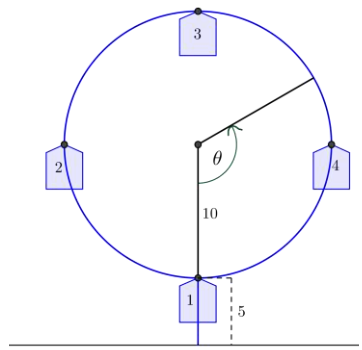
2. Find a formula for a function that models the height of car 1 with respect to time as the kiddie wheel rotates.

3. Is your function in Question 2 the only function that models this situation? Explain how you know.

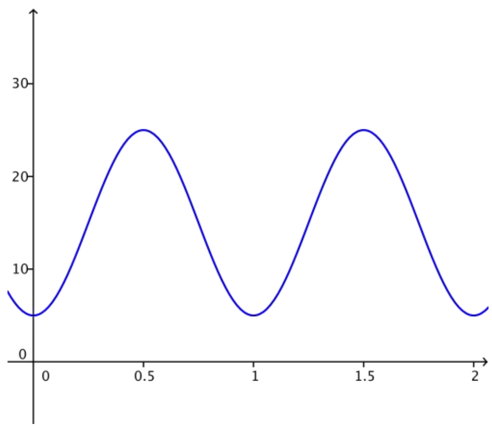
Exit Ticket Sample Solutions

The Ferris Wheel Again

In an amusement park, there is a small Ferris wheel, called a kiddie wheel, for toddlers. The points on the circle in the diagram to the right represent the position of the cars on the wheel. The kiddie wheel has four cars, makes one revolution every minute, and has a diameter of 20 feet. The distance from the ground to a car at the lowest point is 5 feet. Assume $t = 0$ corresponds to a time when car 1 is closest to the ground.



1. **Sketch the height function for car 1 with respect to time as the Ferris wheel rotates for two minutes.**



MP.4

2. **Find a formula for a function that models the height of car 1 with respect to time as the kiddie wheel rotates.**

The horizontal shift is $h = \frac{1}{4}$, the amplitude is 10, and the equation for the midline is $y = 15$. Since the wheel makes one revolution every minute, the period of this function will be 1. Thus, $\omega = \frac{2\pi}{1} = 2\pi$.

$$H(t) = 10\sin\left(2\pi\left(t - \frac{1}{4}\right)\right) + 15$$

MP.3

3. **Is your function in Question 2 the only function that models this situation? Explain how you know.**

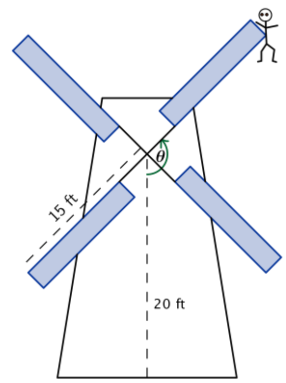
No, any phenomenon that we can model with a sine function can also be modeled with a cosine function using an appropriate horizontal shift and/or reflection about the horizontal axis. Other functions include

$$H(t) = -10\cos(2\pi t) + 15 \text{ or } H(t) = 10\cos\left(2\pi\left(t - \frac{1}{2}\right)\right) + 15.$$

A sine function with a different combination of horizontal translations and reflections could also work.

Problem Set Sample Solutions

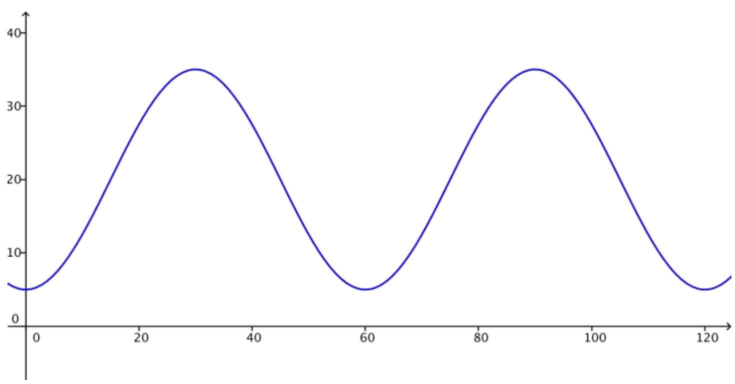
1. In the classic novel *Don Quixote*, the title character famously battles a windmill. In this problem, you will model what happens when Don Quixote battles a windmill, and the windmill wins. Suppose the center of the windmill is 20 feet off the ground, and the sails are 15 feet long. Don Quixote is caught on a tip of one of the sails. The sails are turning at a rate of one counterclockwise rotation every 60 seconds.



   a. Explain why a sinusoidal function could be used to model Don Quixote's height above the ground as a function of time.

   Since Don Quixote is moving around a circle, and the sine function measures vertical displacement of an object moving around a circle, we can use a sine function to track his position above and below the horizontal line through the center of the sails of the windmill. Using transformations, we can use this function to model the height of any point on any circle.

   b. Sketch a graph of Don Quixote's height above the ground as a function of time. Assume $t = 0$ corresponds to a time when he was closest to the ground. What are the amplitude, period, and midline of the graph?



   The amplitude is 15 feet; the period is $\frac{2\pi}{60}$, which is the same as $\frac{\pi}{30}$; and the midline has equation $y = 20$.

MP.4

   c. Model Don Quixote's height H above the ground as a function of time t since he was closest to the ground.

   The height of Don Quixote after t seconds is

   $$H(t) = 15\sin\left(\frac{\pi}{30}(t - 15)\right) + 20.$$

d. After 1 minute and 40 seconds, Don Quixote fell off the sail and straight down to the ground. How far did he fall?

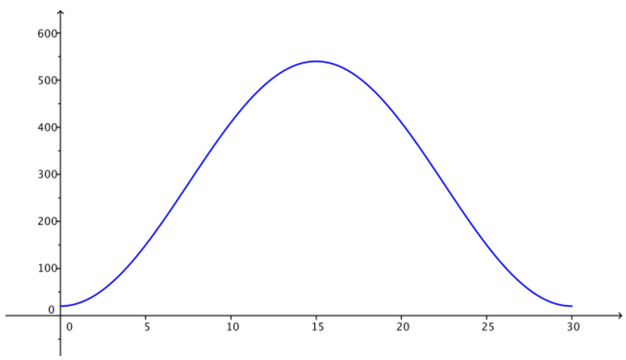
He fell at $t = 100$ seconds. The vertical distance in feet that he fell is the following:

$$\begin{aligned} H(100) &= 15\sin\left(\frac{\pi}{30}(100-15)\right) + 20 \\ &= 15\sin\left(2\pi + \frac{5\pi}{6}\right) + 20 \\ &= 15\sin\left(\frac{5\pi}{6}\right) + 20 \\ &= \frac{15}{2} + 20 \\ &= \frac{55}{2}. \end{aligned}$$

Therefore, $H(100) = 27.5$. Don Quixote fell 27.5 feet.

2. The High Roller, a Ferris wheel in Las Vegas, Nevada, opened in March 2014. The 550 ft. tall wheel has a diameter of 520 feet. A ride on one of its 28 passenger cars lasts 30 minutes, the time it takes the wheel to complete one full rotation. Riders board the passenger cars at the bottom of the wheel. Assume that once the wheel is in motion, it maintains a constant speed for the 30-minute ride and is rotating in a counterclockwise direction.

a. Sketch a graph of the height of a passenger car on the High Roller as a function of the time the ride began.



b. Write a sinusoidal function H that represents the height of a passenger car t minutes after the ride begins.

$$H(t) = 260\sin\left(\frac{2\pi}{30}(t-7.5)\right) + 280$$

c. Explain how the parameters of your sinusoidal function relate to the situation.

The amplitude, 260, is the radius of the wheel. The period is 30 minutes, so $\omega = \frac{2\pi}{30}$. The midline is $y = 280$, which represents how high the center of the wheel is above the ground. To make the graph have a minimal point at $t = 0$, the phase shift can be 7.5, which corresponds to one-fourth of the period.

d. If you were on this ride, how high would you be above the ground after 20 minutes?

$$H(20) = 260\sin\left(\frac{2\pi}{30}(20-7.5)\right) + 280$$

$$H(20) = 410$$

You are 410 feet above the ground.

e. Suppose the ride costs $\$25$. How much does 1 minute of riding time cost? How much does 1 foot of riding distance cost? How much does 1 foot of height above the ground cost?

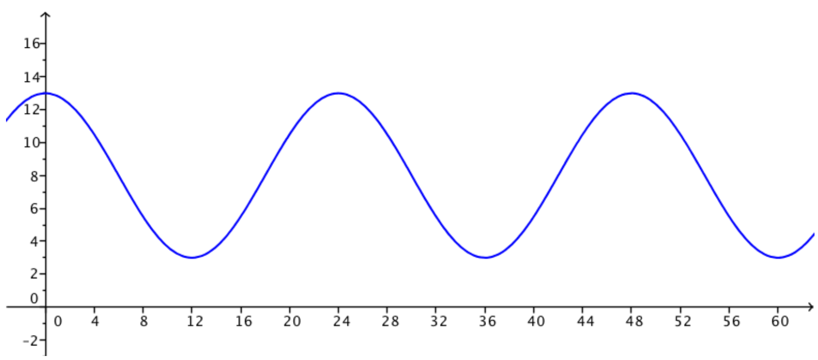
Because $\frac{25}{30} = \frac{5}{6} \approx 0.83$, one minute of riding time costs roughly 83 cents. The circumference of the circle is 520π ft. Dividing 25 by the circumference indicates that each 1 foot of ride costs about 1.5 cents. The height of the wheel is 550 feet. Dividing 25 by 550 indicates that each 1 foot above the ground costs about 4.5 cents.

f. What are some of the limitations of this model based on the assumptions that we made?

We are assuming that the wheel does not stop to add additional passengers when each car gets to the bottom of the wheel during the ride. We are neglecting the fact that the ride starts from a resting position, so its initial speed would be zero. It would take some time to get up to a constant speed once the ride started and slow down to a stop at the end of the ride. We are measuring the height where the passenger car connects to the wheel. In reality, a person on the ride might be slightly higher or lower than this point depending on the design.

3. Once in motion, a pendulum's distance varies sinusoidally from 12 feet to 2 feet away from a wall every 12 seconds.

a. Sketch the pendulum's distance D from the wall over a 1-minute interval as a function of time t. Assume $t = 0$ corresponds to a time when the pendulum was furthest from the wall.



b. Write a sinusoidal function for D, the pendulum's distance from the wall, as a function of the time since it was furthest from the wall.

$$D(t) = 5\cos\left(\frac{2\pi}{24}t\right) + 8$$

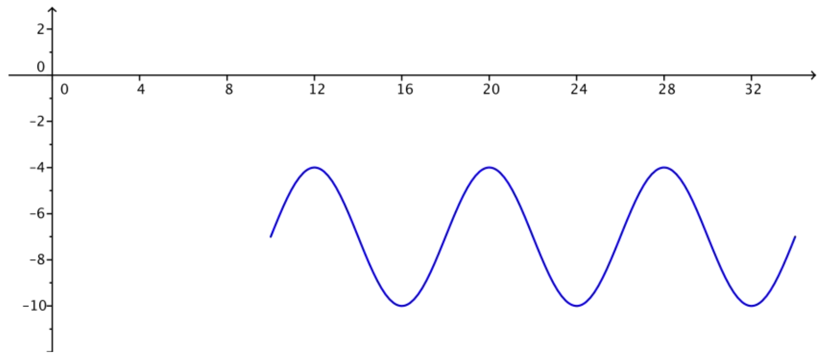
c. Identify two different times when the pendulum was 10 feet away from the wall. (Hint: Write an equation, and solve it graphically.)

$$5\cos\left(\frac{2\pi}{24}t\right) + 8 = 10$$

The first two positive solutions to this equation are 4.47 seconds and 19.57 seconds.

4. The height in meters relative to the starting platform height of a car on a portion of a roller coaster track is modeled by the function $H(t) = 3\sin\left(\frac{\pi}{4}(t-10)\right) - 7$. It takes a car 24 seconds to travel on this portion of the track, which starts 10 seconds into the ride.

   a. Graph the height relative to the starting platform as a function of time over this time interval.



   b. Explain the meaning of each parameter in the height function in terms of the situation.

   The amplitude 3 means that the height ranges vertically a total distance of 6 feet. The graph completes one cycle every 8 seconds because the period is 8. The phase shift of 10 and the midline equation of $y = -7$ indicate that this car is 7 feet below the platform after 10 seconds, and at this point, the car is ascending.

5. Given the following function, use the parameters to formulate a real-world situation involving one dimension of circular motion that could be modeled using this function. Explain how each parameter of the function relates to your situation.

$$f(x) = 10\sin\left(\frac{\pi}{8}(x-3)\right) + 15$$

Student scenarios will vary; however, f should represent a horizontal or vertical distance from a reference point as a point is in motion around a circle, and x should be an amount of rotation or a time (assuming constant rotation speed). The radius of the circle should be 10. The object should rotate around the circle once every 16 units. When $x = 3$, the object's distance from the reference point should be 15 units, and its distance from the reference point should be increasing.

Lesson 13: Tides, Sound Waves, and Stock Markets

Student Outcomes

- Students model cyclical phenomena from biological and physical science using trigonometric functions.
- Students understand that some periodic behavior is too complicated to be modeled by simple trigonometric functions.

Lesson Notes

Together, the examples in the lesson help students see the ways in which some periodic data can be modeled by sinusoidal functions. These examples offer students the opportunity to solve real-world problems (MP.1) to experience ways in which modeling data with sinusoidal functions is valuable. Each example supports the main objective of exploring the usefulness and limitations of sinusoidal functions using the most well-known contexts of such functions. Throughout the lesson, students employ MP.4 in creating models of real-world contexts. Additionally, these models offer opportunities to engage in MP.2, MP.5, and MP.6.

In Lessons 11 and 12, students found sinusoidal equations to model phenomena that were clearly periodic, such as the height of a moving passenger car on a Ferris wheel. The focus of this lesson is on fitting a function to given data, as specified by standard **S-ID.B.6a**. Students are asked to find functions that could be used to model data that appear to be periodic in nature, requiring them to make multiple choices about approximating the amplitude, period, and midline of a graph that approximates the data points. As a result, there are multiple correct responses based on how students choose to model the data. Be sure to accept and discuss these different results because this is an important part of the modeling process. If two students (or groups of students) create different formulas, that does not necessarily mean that one of them made a mistake; they both could have created valid, though different, models.

There are many options for incorporating technology into this particular lesson. Graphing calculators or online graphing programs such as Desmos can help students create a scatter plot very quickly. On a graphing calculator, students can enter the data into lists, make a scatter plot, and then use the sinusoidal regression feature to determine the equation of a function that fits the data. It is best to wait until Example 2 to use calculator regression to fit data with a sinusoidal function, with students fitting the tidal data in Example 1 with a sinusoidal function by hand. To save time, preload the data into the calculators, or have students enter the data on their own the night before. If students are using online graphing tools such as Desmos, they can enter the data into a table, graph it, and then write the equation to check that it corresponds to their graph. The teacher can also create scatter plots quickly in a spreadsheet. Depending on students' familiarity with technology, this lesson may take more or less time than indicated.

Opening Exercise (5 minutes)

Briefly review the text that begins this lesson by doing a close read. Have students read the section once independently and then again as the teacher or a volunteer reads the passage aloud. While the passage is read aloud, have students underline any important information, such as the meaning of the MLLW. Check for understanding of this term by asking students to explain to a partner why the height at 9:46 a.m. on Wednesday was -0.02 feet and then having several students report out briefly to the entire class.

MP.5 & MP.6

The teacher may choose to allow students to use graphing technology to create their scatter plots. In any case, students need access to graph paper for accurately recording a scatter plot. If working in small groups, each group could create the scatter plot on a piece of gridded chart paper as well. Watch carefully as students scale their graphs and translate the times from the table to the time since midnight on May 21. For example, the first time at 2:47 a.m. would be $2 + \frac{47}{60} \approx 2.8$ hours since midnight. Encourage students to discuss what would be an appropriate level of precision to get an accurate representation of the tidal data. If groups make different decisions regarding precision of the time measurement, then the teacher can highlight the slight differences in their graphs and functions at the end of Example 1.

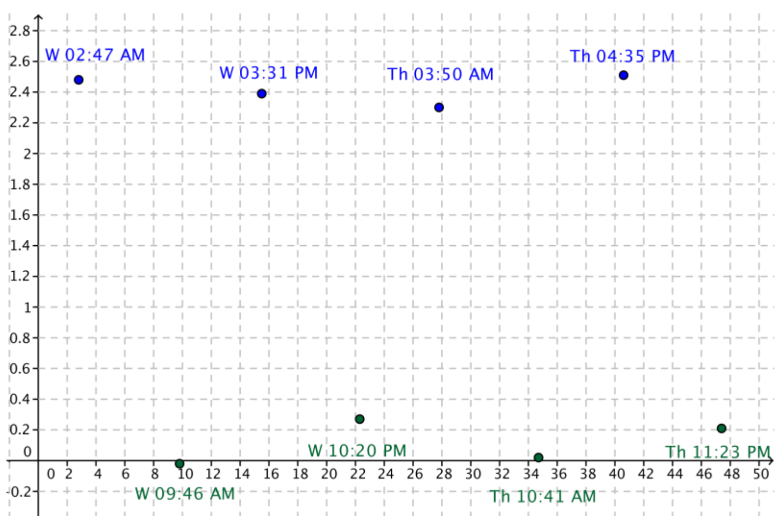
Opening Exercise

Anyone who works on or around the ocean needs to have information about the changing tides to conduct their business safely and effectively. People who go to the beach or out onto the ocean for recreational purposes also need information about tides when planning their trip. The table below shows tide data for Montauk, NY, for May 21–22, 2014. The heights reported in the table are relative to the Mean Lower Low Water (MLLW). The MLLW is the average height of the lowest tide recorded at a tide station each day during the recording period. This reference point is used by the National Oceanic and Atmospheric Administration (NOAA) for the purposes of reporting tidal data throughout the United States. Each different tide station throughout the United States has its own MLLW. High and low tide levels are reported relative to this number. Since it is an average, some low tide values can be negative. NOAA resets the MLLW values approximately every 20 years.

MONTAUK, NY, TIDE CHART

Date	Day	Time	Height in Feet	High/Low
2014/05/21	Wed.	02:47 a.m.	2.48	H
2014/05/21	Wed.	09:46 a.m.	-0.02	L
2014/05/21	Wed.	03:31 p.m.	2.39	H
2014/05/21	Wed.	10:20 p.m.	0.27	L
2014/05/22	Thurs.	03:50 a.m.	2.30	H
2014/05/22	Thurs.	10:41 a.m.	0.02	L
2014/05/22	Thurs.	04:35 p.m.	2.51	H
2014/05/22	Thurs.	11:23 p.m.	0.21	L

a. **Create a scatter plot of the data with the horizontal axis representing time since midnight on May 21 and the vertical axis representing the height in feet relative to the MLLW.**



Scaffolding:

For struggling students:

- Provide graph paper with the axes and scales already labeled.
- Model the process of converting times in hours and minutes to decimals.

b. **What type of function would best model this set of data? Explain your choice.**

Even though the maximum points do not all have the same value, a sinusoidal function would best model this data because the data repeat the pattern of high point, low point, high point, low point, over fairly regular intervals of time—roughly every 6 hours.

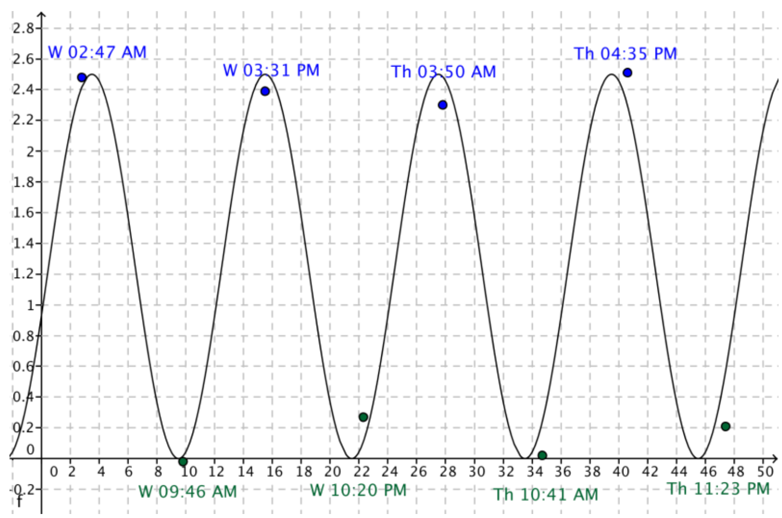
Example 1 (5 minutes): Write a Sinusoidal Function to Model a Set of Data

Tidal data do not lie perfectly on a sinusoidal curve, but the heights vary in a predictable pattern. The tidal data chosen for this example appear to be very sinusoidal. Investigating tide charts (which often include graphical representations) for different locations reveals a great deal of variation. Depending on the location where the height of the tide is measured, the cycle of the moon, and the season, tides vary from one cycle to the next. The intent of the Opening Exercise and this example is to use a sinusoidal function as a model for the tides and to recognize the limitations of this type of model when students are presented with additional data.

Depending on their skill level, students can work through this problem in small groups, or the teacher can use a more direct approach to demonstrate how to identify the key features of the graph of a sinusoid, which are used to determine the parameters in the corresponding function. The difference between this example and previous examples is that the data here are not perfect. Students have to make some decisions about where to place the midline, amplitude, and period of the graph. If demonstrating this for the class, consider projecting an image of the scatter plot on a white board, having a hand-drawn graph on the board, or working on a sheet of chart paper.

Example 1: Write a Sinusoidal Function to Model a Set of Data

a. **On the scatter plot you created in the Opening Exercise, sketch the graph of a sinusoidal function that could be used to model the data.**



b. **What are the midline, amplitude, and period of the graph?**

Answers will vary depending on student graphs but should be close to those listed below.

Midline: $y = 1.25$

Amplitude: 1.25

Period: 12

c. **Estimate the horizontal distance between a point on the graph that crosses the midline and the vertical axis.**

The distance is about 12.5 hours.

d. **Write a function of the form $f(x) = A\sin(\omega(x-h)) + k$ to model these data, where x is the hours since midnight on May 21, and $f(x)$ is the height in feet relative to the MLLW.**

$f(x) = 1.25\sin\left(\frac{2\pi}{12}(x-12.5)\right) + 1.25.$

Discussion (5 minutes)

MP.2

Use this brief discussion to clarify the relationship between the features on the graph and the parameters in the sinusoidal function. Students likely struggle most with recalling that the period is *not* equal to ω; rather, $|\omega| = \frac{2\pi}{P}$, where P is the period of the function. Another difficulty is determining an appropriate value for h. Since this is a periodic function, several values would work; the simplest way is to select the first instance when the graph is near the midline and increasing. Then, the value of h represents a horizontal translation of the function when A is even more critical to allow students to fully process how the features of the graph and the parameters in the function are related.

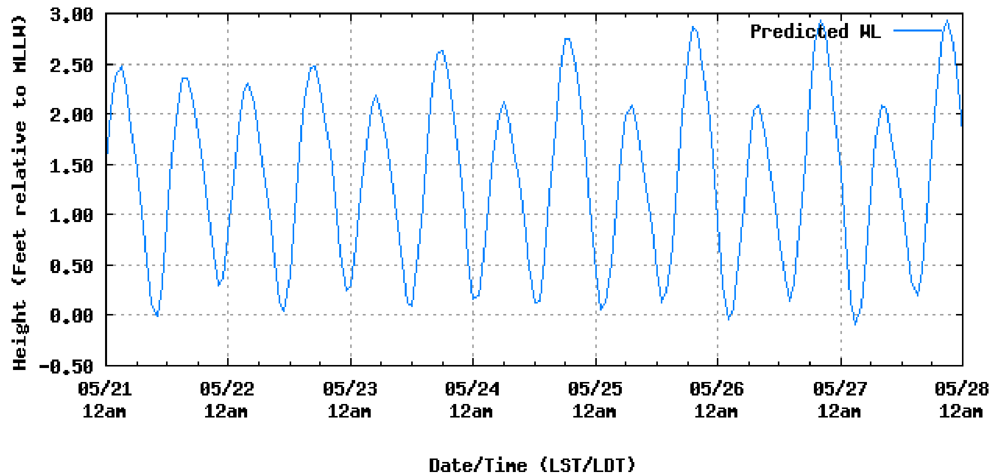
- How do the answers to parts (b) and (c) relate to the parameters in the function you wrote?
  - *The length $|A|$ is the amplitude, $y = k$ is the midline, h is the distance the graph of $f(x) = A\sin(\omega(x))$ was translated horizontally, and ω is related to the frequency and period. The graph of f is the image of the graph of the sine function after a vertical scaling by the factor A and horizontal scaling by a factor $\frac{1}{\omega}$ and translating horizontally by h and vertically by k units.*
- How do parameters in the function relate to the tides at Montauk?
  - *The difference between the maximum value and minimum value of the function gives the fluctuation in the tides. The maximum value is the height of the high tide, and the minimum value is the height of the low tide relative to the MLLW. These height variations are less than 2.5 ft. If we take half of the period, we can estimate the time between the high and low tides.*

Exercise 1 (5 minutes)

Give students an opportunity to respond to this exercise in small groups; then, have groups share their findings with the whole class. Close this section of the lesson with a whole class discussion around the questions that appear below. Have students discuss their responses with a partner before calling on one or two students to answer the questions with the entire class. Students who are familiar with ocean tides may adapt more readily to the variation in tidal heights and times. Encourage students to look up the data for the tides at Montauk on the day this lesson is taught to see how they compare to the data given here.

Exercise 1

1. **The graph of the tides at Montauk for the week of May 21–28 is shown below. How accurately does your model predict the time and height of the high tide on May 28?**



Source: http://tidesandcurrents.noaa.gov/

The maximum value of f is 2.5. The high tide on May 28 was almost 3 feet above MLLW. The model is off by approximately 6 inches.

- Are tides an example of periodic phenomena? Why do you suppose this is true?
  - *Yes, to some degree, but they are not perfectly modeled by a sinusoidal function. Tides are influenced by the moon's gravitational field. Because the moon orbits Earth at a regular interval of time, the influence of the moon on tidal heights would be periodic up to a point since there will be ongoing variation in heights due to a wide variety of natural factors.*
- What are the limitations of using a sinusoidal function to model the height of tides over a long period of time at a particular location?
  - *At different times during the year, the height of a tide varies widely. Additionally, not all tides are perfectly symmetric.*

In the Exit Ticket, students examine tidal data near New Orleans, Louisiana. The data for one location has two different maximum and minimum values, so they would definitely not be modeled by a sinusoidal function. Emphasize to students that mathematical models that they create are often only valid for the set of data provided and may or may not be good predictors of future or past behavior.

Example 2 (5 minutes): Digital Sampling of Sound

When someone hears a musical note, her ear drums are sensing pressure waves that are modeled by sinusoids, or combinations of sinusoids. As the amplitude of the wave increases, the note sounds louder. As the frequency of the wave increases (which means the period is decreasing), the pitch of the note increases. A pure tone of the note A has a frequency of 440 Hz, which means the graph of the sinusoidal function that represents this pure tone would complete 440 cycles in one second. This function would have a period of $\frac{2\pi}{440}$ seconds. If the sampling rate increases, the sound quality improves. To avoid distortion of the sound that is perceptible to the human ear, a sampling rate must be used that is more than twice the frequency of the tone. If the resolution increases (represented by the vertical scaling), the quality of the digital sample also improves. In reality, analog sounds are converted to digital sounds at a sampling rate of thousands of times per second. A typical resolution is 16 bit, which means the difference between the highest and lowest values are be divided into 2^{16} equal sections, and the actual tone is assigned to the closest of those values at each sampling point. This example is simplified, using a much lower sampling rate and resolution to allow students to understand the concept and keep the total number of data points manageable.

Be sure to include at least part (a) of this example. If time permits, students can enter the two sets of data into a graphing calculator and use the sinusoidal regression feature of the graphing calculator to create the function that fits the sampled data points. Have students compare and contrast the actual parameters of the actual function graphed below and the parameters in the models generated by their graphing calculator. Alternately, present the three equations as directed below, and then ask students to compare those parameters to confirm or refute their conjectures in part (a).

Example 2: Digital Sampling of Sound

When sound is recorded or transmitted electronically, the continuous waveform is sampled to convert it to a discrete digital sequence. If the sampling rate (represented by the horizontal scaling) or the resolution (represented by the vertical scaling) increases, the sound quality of the recording or transmission improves.

The data graphed below represent two different samples of a pure tone. One is sampled 8 times per unit of time at a 2-bit resolution (4 equal intervals in the vertical direction), and the other is sampled 16 times per unit of time at a 4-bit resolution (8 equal intervals in the vertical direction).

a. Which sample points would produce a better model of the actual sound wave? Explain your reasoning.

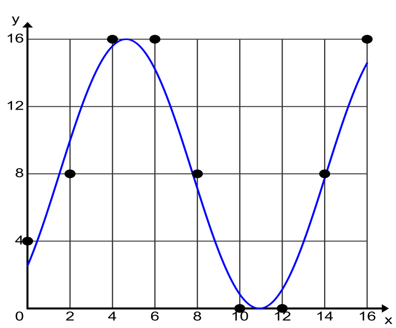


Figure 1: Sampled 8 times per unit of time, 2-bit resolution.

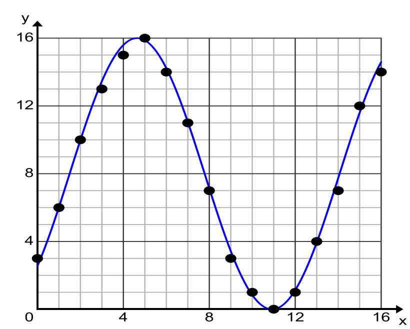


Figure 2: Sampled 16 times per unit of time, 4-bit resolution.

You will get a better model with the sample points in Figure 2. If the function we use to model the data passes through more data points on the original sound wave, the curve we use to fit the data will accurately represent the original sound.

b. Find an equation for a function g that models the pure tone sampled in Figure 1.

$g(x) = 8.6436\sin(0.5089x - 0.8891) + 8.3651$

c. Find an equation for a function h that models the pure tone sampled in Figure 2.

$h(x) = 7.8464\sin(0.4981x - 0.7327) + 7.8760$

If students do not create the models based on the data points due to time constraints, you can share the actual sinusoidal function $f(x) = 8\sin\big(0.5(x - 1.5)\big) + 8$ and the models for each case with the class and have them compare and contrast the parameters in each function. If time permits, students can graph all three functions to visually illustrate the differences and begin to understand why a digital sample is not exactly the same as the pure tone. Note that making use of available technology to create sinusoidal functions to fit the data and to graph the functions greatly affects the timing of this lesson.

Exercises 2–6 (10 minutes)

In these exercises, students explore data that could be modeled by the sum of two functions and are asked to consider what happens to the validity of this model over a longer time period. These data lie on a curve that shows the characteristic oscillation of a sinusoidal function but also model the historic trend that stock prices increase over time. Given the financial crisis in 2008 and the Great Recession, even an algebraic function that takes into account steady increases over time would not accurately predict future stock values during that time period. As you debrief these questions, take time to again reinforce that models have limitations, especially when many factors are contributing to the variability of the quantities involved in any given situation. Note that these are actual data from MSFT (Microsoft) stock.

Exercises 2–6

Stock prices have historically increased over time, but they also vary in a cyclical fashion. Create a scatter plot of the data for the monthly stock price for a 15-month time period since January 1, 2003.

Months Since Jan. 1, 2003	Price at Close in dollars
0	20.24
1	19.42
2	18.25
3	19.13
4	19.20
5	20.91
6	20.86
7	20.04
8	20.30
9	20.54
10	21.94
11	21.87
12	21.51
13	20.65
14	19.84

2. **Would a sinusoidal function be an appropriate model for these data? Explain your reasoning.**

   A sinusoidal function would not be appropriate because the data are trending upward as time passes.

We can model the slight upward trend in these data with the linear function $f(x) = 19.5 + 0.13x$.

If we add a sinusoidal function to this linear function, we can model these data with an algebraic function that displays an upward trend but also varies in a cyclical fashion.

3. Find a sinusoidal function, g, that when added to the linear function, f, models these data.

$g(x) = 1.1\sin\left(\frac{2\pi}{5.5}(x-4)\right)$ *(Student answers may vary.)*

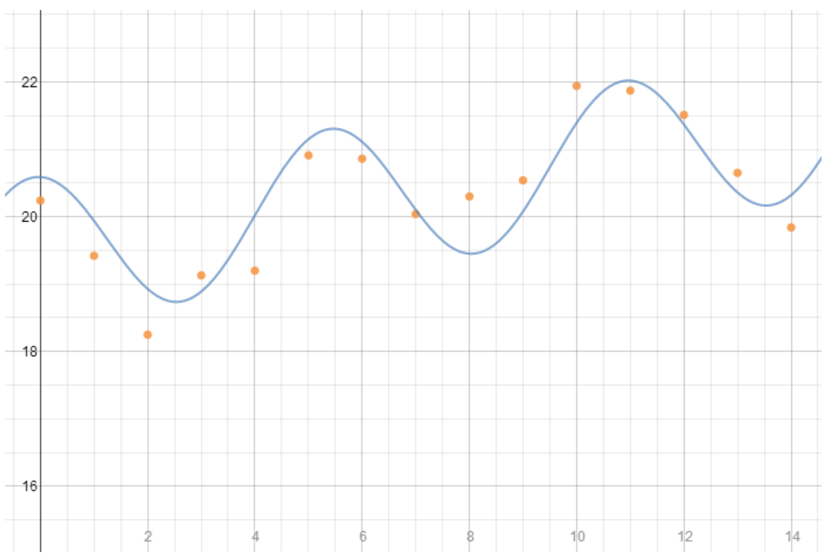
4. Let S be the stock price function that is the sum of the linear function listed above and the sinusoidal function in Exercise 3.

$S(x) =$ ________________________________.

$S(x) = 19.5 + 0.13x + 1.1\sin\left(\frac{2\pi}{5.5}(x-4)\right)$

5. Add the graph of this function to your scatter plot. How well does it appear to fit the data?

The scatter plot with the function S is shown below for the first 15 months. In general, it appears to model these data, but there are a few outliers such as the stock value two months after data were reported.



After students complete Exercises 2–5, share the extended graph of the data with them and have them respond in small groups to the last exercise. Conclude by briefly discussing the financial crisis in 2008–2009. Have students who are interested plot the additional data and compare the price their model would predict to the value of the stock in 2013. This particular stock recovered quite well from the financial crisis. Additional data are provided below for students wishing to extend their work on this problem.

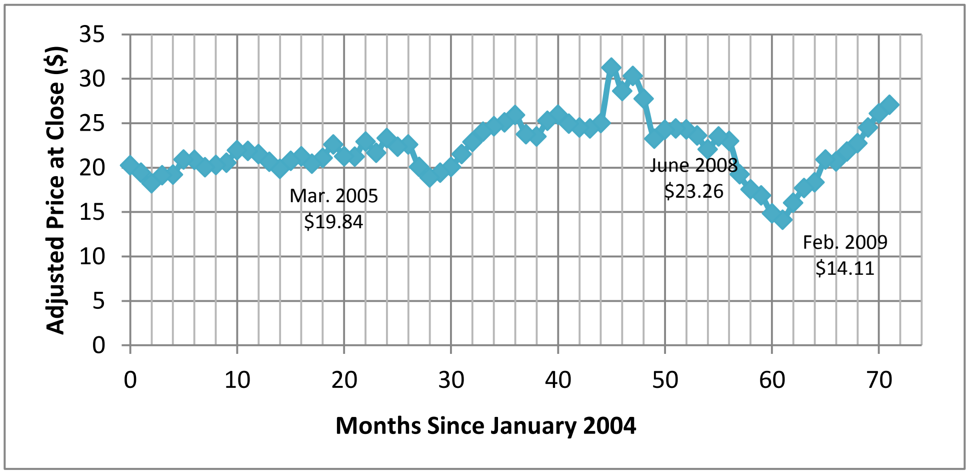
Months Since Jan. 2003	Adjusted Price at Close ($)	Date
119	36.87	12/2/2013
118	37.58	11/1/2013
117	34.64	10/1/2013
116	32.55	9/3/2013
115	32.67	8/1/2013
114	30.93	7/1/2013
113	33.55	6/3/2013
112	33.90	5/1/2013
111	31.93	4/1/2013
110	27.60	3/1/2013
109	26.82	2/1/2013
108	26.26	1/2/2013
107	25.55	12/3/2012
106	25.47	11/1/2012
105	27.08	10/1/2012
104	28.24	9/4/2012
103	29.24	8/1/2012
102	27.78	7/2/2012
101	28.83	6/1/2012
100	27.51	5/1/2012
99	29.99	4/2/2012
98	30.21	3/1/2012
97	29.72	2/1/2012
96	27.47	1/3/2012

Months Since Jan. 2003	Adjusted Price at Close ($)	Date
95	24.15	12/1/2011
94	23.80	11/1/2011
93	24.59	10/3/2011
92	22.98	9/1/2011
91	24.56	8/1/2011
90	25.14	7/1/2011
89	23.86	6/1/2011
88	22.95	5/2/2011
87	23.63	4/1/2011
86	23.15	3/1/2011
85	24.23	2/1/2011
84	25.13	1/3/2011
83	25.29	12/1/2010
82	22.89	11/1/2010
81	24.02	10/1/2010
80	22.06	9/1/2010
79	21.14	8/2/2010
78	23.12	7/1/2010
77	20.62	6/1/2010
76	23.12	5/3/2010
75	27.24	4/1/2010
74	26.12	3/1/2010
73	25.57	2/1/2010
72	25.02	1/4/2010

Months Since Jan. 2003	Adjusted Price at Close ($)	Date
71	27.06	12/1/2009
70	26.11	11/2/2009
69	24.51	10/1/2009
68	22.73	9/1/2009
67	21.79	8/3/2009
66	20.67	7/1/2009
65	20.89	6/1/2009
64	18.36	5/1/2009
63	17.69	4/1/2009
62	16.04	3/2/2009
61	14.11	2/2/2009
60	14.83	1/2/2009
59	16.86	12/1/2008
58	17.54	11/3/2008
57	19.24	10/1/2008
56	23.00	9/2/2008
55	23.51	8/1/2008
54	22.07	7/1/2008
53	23.61	6/2/2008
52	24.30	5/1/2008
51	24.39	4/1/2008
50	24.27	3/3/2008
49	23.26	2/1/2008
48	27.77	1/2/2008
47	30.32	12/3/2007
46	28.62	11/1/2007
45	31.25	10/1/2007
44	25.01	9/4/2007
43	24.39	8/1/2007

Months Since Jan. 2003	Adjusted Price at Close ($)	Date
42	24.52	7/2/2007
41	24.93	6/1/2007
40	25.96	5/1/2007
39	25.25	4/2/2007
38	23.5	3/1/2007
37	23.75	2/1/2007
36	25.93	1/3/2007
35	25.09	12/1/2006
34	24.67	11/1/2006
33	24.04	10/2/2006
32	22.90	9/1/2006
31	21.52	8/1/2006
30	20.07	7/3/2006
29	19.44	6/1/2006
28	18.90	5/1/2006
27	20.07	4/3/2006
26	22.61	3/1/2006
25	22.33	2/1/2006
24	23.32	1/3/2006
23	21.66	12/1/2005
22	22.93	11/1/2005
21	21.23	10/3/2005
20	21.25	9/1/2005
19	22.61	8/1/2005
18	21.09	7/1/2005
17	20.45	6/1/2005
16	21.25	5/2/2005
15	20.77	4/1/2005

6. **Here is a graph of the same stock through December 2009.**



a. **Will your model do a good job of predicting stock values past the year 2005?**

It will not do a very good job of predicting the value of the stock because it rises sharply and then plummets before starting to rise in price after a low in 2009.

b. **What event occurred in 2008 to account for the sharp decline in the value of stocks?**

There was a financial crisis in the stock markets that marked the start of a recession.

c. **What are the limitations of using any function to make predictions regarding the value of a stock at some point in the future?**

There are many variables that can affect the price of a stock. A function that is based only on time will be limited in its ability to predict future events during unforeseen circumstances.

Closing (5 minutes)

Use the discussion questions to close this lesson. Have students respond individually or share with a partner before discussing them as a whole class. These questions offer an opportunity to determine whether students understand the benefits as well as the inherent limitations of using mathematical functions to represent real-world situations.

- How confident are you that mathematical models can help us make predictions about future values of a particular quantity?
- When creating a function to represent a data set, what are some of the limitations in using this function?
- Mathematical models are used frequently to represent real-world situations like these. Even given the limitations, why would scientists and economists find it useful to have a function to represent the relationship between the data they are studying?

Finally, ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

Periodic data can be modeled with either a sine or a cosine function by extrapolating values of the parameters A, ω, h, and k from the data and defining a function $f(t) = A\sin(\omega(t-h)) + k$ or $g(t) = A\cos(\omega(t-h)) + k$, as appropriate.

Sine or cosine functions may not perfectly fit most data sets from actual measurements; therefore, there are often multiple functions used to model a data set.

If possible, plot the data together with the function that appears to fit the graph. If it is not a good fit, adjust the model and try again.

Exit Ticket (5 minutes)

Name ______________________________ Date ____________________

Lesson 13: Tides, Sound Waves, and Stock Markets

Exit Ticket

Tidal data for New Canal Station, located on the shore of Lake Pontchartrain, LA, and Lake Charles, LA, are shown below.

New Canal Station on Lake Pontchartrain, LA, Tide Chart

Date	Day	Time	Height	High/Low
2014/05/28	Wed.	07:22 a.m.	0.12	L
2014/05/28	Wed.	07:11 p.m.	0.53	H
2014/05/29	Thurs.	07:51 a.m.	0.11	L
2014/05/29	Thurs.	07:58 p.m.	0.53	H

Lake Charles, LA, Tide Chart

Date	Day	Time	Height	High/Low
2014/05/28	Wed.	02:20 a.m.	-0.05	L
2014/05/28	Wed.	10:00 a.m.	1.30	H
2014/05/28	Wed.	03:36 p.m.	0.98	L
2014/05/28	Wed.	07:05 p.m.	1.11	H
2014/05/29	Thurs.	02:53 a.m.	-0.06	L
2014/05/29	Thurs.	10:44 a.m.	1.31	H
2014/05/29	Thurs.	04:23 p.m.	1.00	L
2014/05/29	Thurs.	07:37 p.m.	1.10	H

1. Would a sinusoidal function of the form $f(x) = A\sin\big(\omega(x-h)\big) + k$ be appropriate to model the given data for each location? Explain your reasoning.

2. Write a sinusoidal function to model the data for New Canal Station.

Exit Ticket Sample Solutions

Tidal data for New Canal Station, located on the shore of Lake Pontchartrain, LA, and Lake Charles, LA, are shown below.

New Canal Station on Lake Pontchartrain, LA, Tide Chart

Date	Day	Time	Height	High/Low
2014/05/28	Wed.	07:22 a.m.	0.12	L
2014/05/28	Wed.	07:11 p.m.	0.53	H
2014/05/29	Thurs.	07:51 a.m.	0.11	L
2014/05/29	Thurs.	07:58 p.m.	0.53	H

Lake Charles, LA, Tide Chart

Date	Day	Time	Height	High/Low
2014/05/28	Wed.	02:20 a.m.	-0.05	L
2014/05/28	Wed.	10:00 a.m.	1.30	H
2014/05/28	Wed.	03:36 p.m.	0.98	L
2014/05/28	Wed.	07:05 p.m.	1.11	H
2014/05/29	Thurs.	02:53 a.m.	-0.06	L
2014/05/29	Thurs.	10:44 a.m.	1.31	H
2014/05/29	Thurs.	04:23 p.m.	1.00	L
2014/05/29	Thurs.	07:37 p.m.	1.10	H

1. **Would a sinusoidal function of the form $f(x) = A\sin(\omega(x-h)) + k$ be appropriate to model the given data for each location? Explain your reasoning.**

   The data for New Canal Station could be modeled with a sinusoidal function, but the other data would not work very well since there appear to be two different low tide values that vary by approximately a foot.

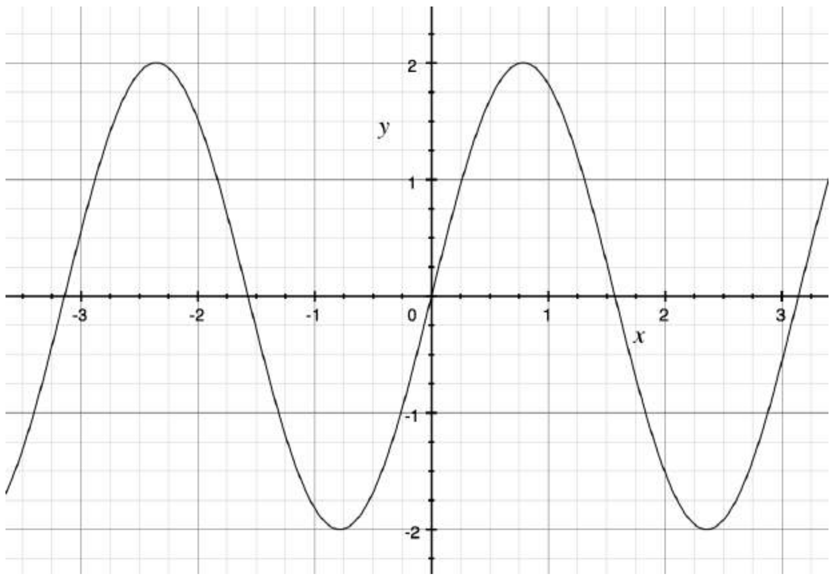
2. **Write a sinusoidal function to model the data for New Canal Station.**

   The amplitude is $\frac{(0.53-0.115)}{2} = 0.2075$. *The period appears to be approximately* 24.5 *hours. The value of* k *that determines the midline is* $\frac{0.53+0.115}{2} = 0.3225$. *Since the graph starts at its lowest value at 7:22 a.m., use a negative cosine function with a horizontal shift of approximately* 7.4.

   $$f(x) = -0.2075\cos\left(\frac{2\pi}{24.5}(x-7.4)\right) + 0.3225$$

Problem Set Sample Solutions

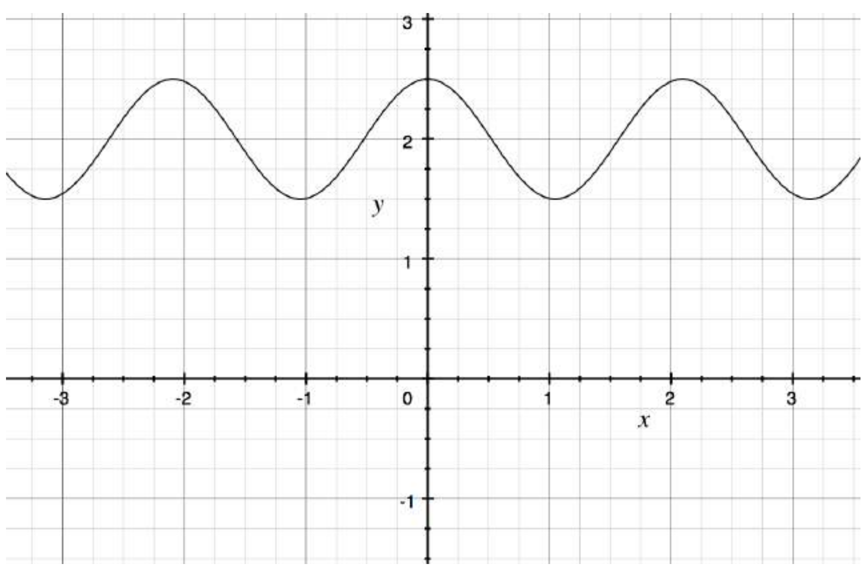
1. Find equations of both a sine function and a cosine function that could each represent the graph given below.



$y = 2\sin(2x)$

$y = 2\cos\big(2(x+\pi)\big)$

2. Find equations of both a sine function and a cosine function that could each represent the graph given below.



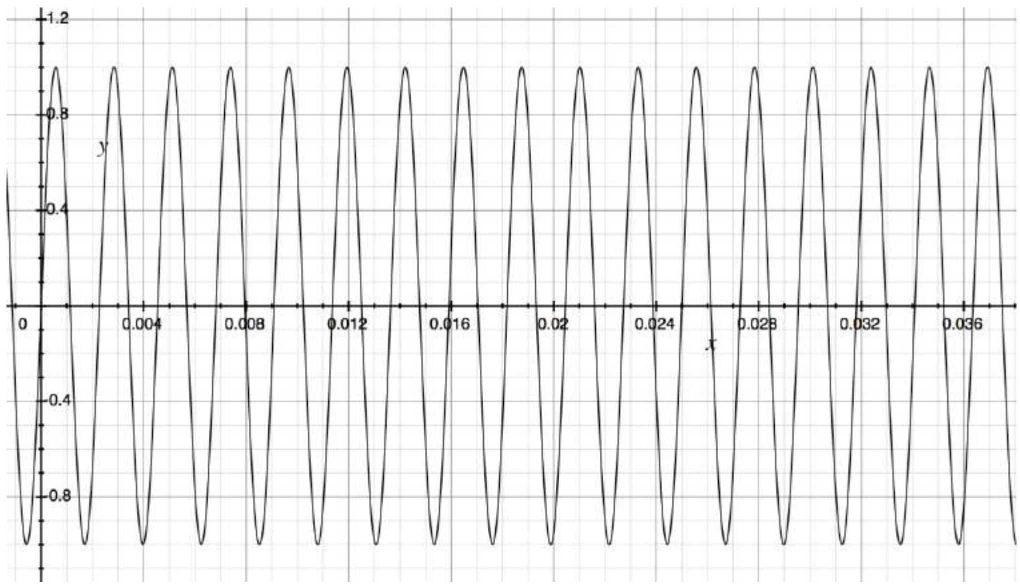
$y = 0.5\cos(3x) + 2$

$y = 0.5\sin\left(3\left(x - \frac{\pi}{2}\right)\right) + 2$

3. Rapidly vibrating objects send pressure waves through the air that are detected by our ears and then interpreted by our brains as sound. Our brains analyze the amplitude and frequency of these pressure waves.

A speaker usually consists of a paper cone attached to an electromagnet. By sending an oscillating electric current through the electromagnet, the paper cone can be made to vibrate. By adjusting the current, the amplitude and frequency of vibrations can be controlled.

The following graph shows the pressure intensity (I) as a function of time (x), in seconds, of the pressure waves emitted by a speaker set to produce a single pure tone.



a. Does it seem more natural to use a sine or a cosine function to fit to this graph?

Either a sine or a cosine function could be used, but since the graph passes through the origin, it is natural to use a sine function.

b. Find the equation of a trigonometric function that fits this graph.

Reading from the graph, we have $A = 1$, $h = 0$, $k = 0$, and $2p \approx 0.0045$ (because the graph has an x-intercept at approximately 0.0045 after two full periods). Then, $p \approx 0.00225$, so $\omega = \frac{2\pi}{p}$ gives $\omega \approx 800\pi$. Thus, an equation that models the graph presented here is $I(x) = \sin(800\pi x)$.

4. Suppose that the following table represents the average monthly ambient air temperature, in degrees Fahrenheit, in some subterranean caverns in southeast Australia for each of the twelve months in a year. We wish to model these data with a trigonometric function. (Notice that the seasons are reversed in the Southern Hemisphere, so January is in summer, and July is in winter.)

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
°F	64.04	64.22	61.88	57.92	53.60	50.36	49.10	49.82	52.34	55.22	58.10	61.52

a. Does it seem reasonable to assume that these data, if extended beyond one year, should be roughly periodic?

Yes

b. **What seems to be the amplitude of the data?**

There are a number of ways to approximate the amplitude. We can simply take half of the difference of the highest and lowest temperatures, or we can take half of the difference of the highest temperature and the one six months later (or half of the difference of the lowest temperature and the one that is six months earlier). The highest temperature is $64.22°\text{F}$*; the lowest temperature is* $49.82°\text{F}$*. Using these temperatures, we approximate the amplitude to be* $\frac{64.22°\text{F} - 49.10°\text{F}}{2} = 7.56°\text{F}$*.*

c. **What seems to be the midline of the data (equation of the line through the middle of the graph)?**

The average of $64.22°\text{F}$ *and* $49.10°\text{F}$ *six months later is* 56.66*. Thus, the midline would be* $y = 56.66$*; so, we have* $k = 56.66$*.*

d. **Would it be easier to use sine or cosine to model these data?**

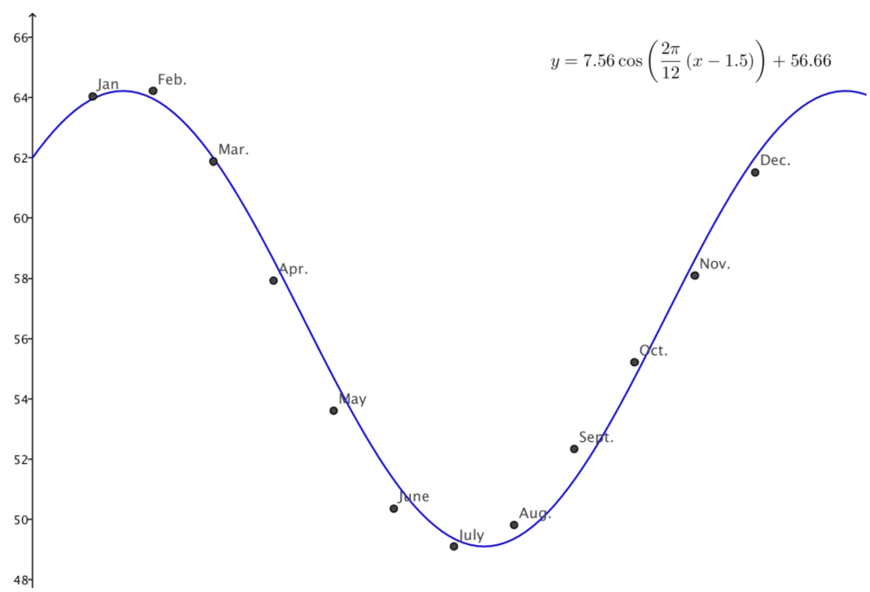
Cosine would be easier because we can pinpoint a peak of the function more easily than where it should cross the midline.

e. **What is a reasonable approximation for the horizontal shift?**

It appears that the peak of the function would be between January and February, so let's use $h = 1.5$ *for the horizontal shift. (This assumes that we have labeled January as month 1 and February as month 2.)*

f. **Write an equation for a function that could fit these data.**

$$F(x) = 7.56\cos\left(\frac{2\pi}{12}(x - 1.5)\right) + 56.66$$



5. The table below provides data for the number of daylight hours as a function of day of the year, where day 1 represents January 1. Graph the data and determine if they could be represented by a sinusoidal function. If they can, determine the period, amplitude, and midline of the function, and find an equation for a function that models the data.

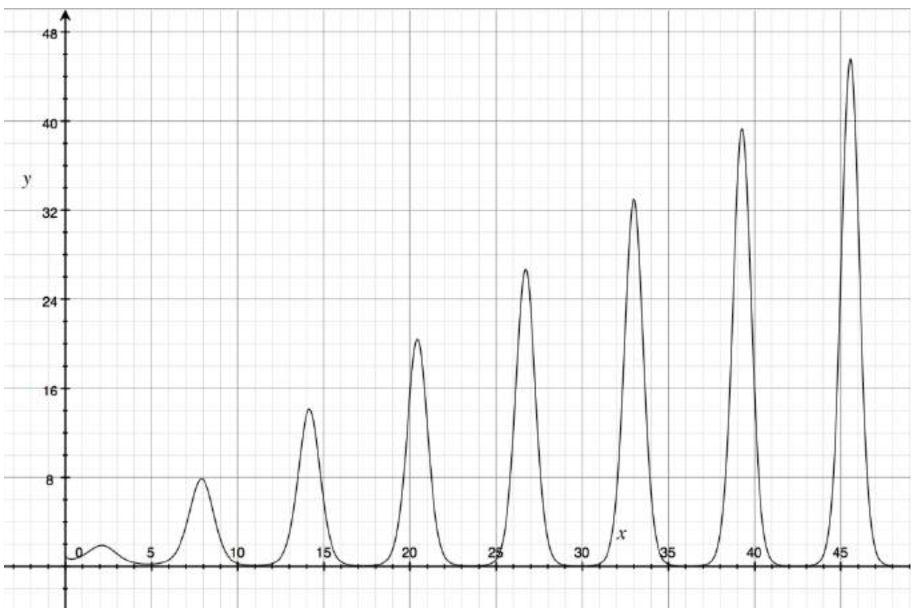
Day of Year	0	50	100	150	175	200	250	300	350
Hours	4.0	7.9	14.9	19.9	20.5	19.5	14.0	7.1	3.6

The data appear sinusoidal, and the easiest function to model them with is a cosine function. The period would be 365 days, so the frequency would be $\omega = \frac{2\pi}{365} \approx 0.017$. The amplitude is $\frac{1}{2}(20.5 - 3.6) = 8.45$, and the midline is $y = k$, where $k = \frac{1}{2}(20.5 + 3.6) = 12.05$. We want the highest value at the peak, so the horizontal shift is 175.

$H = 8.5\cos(0.017t - 175) + 12.05.$

MP.3

6. The function graphed below is $y = x^{\sin(x)}$. Blake says, "The function repeats on a fixed interval, so it must be a sinusoidal function." Respond to his argument.



While the equation for the function includes a sine function within it, the function itself is not a sinusoidal function. It does not have a constant amplitude or midline, though it does appear to become zero at fixed increments; it does not have a period because the function values do not repeat. Thus, it is not a periodic function and is not sinusoidal.

Lesson 14: Graphing the Tangent Function

Student Outcomes

- Students graph the tangent function.
- Students use the unit circle to express the values of the tangent function for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of $\tan(x)$, where x is any real number in the domain of the tangent function.

Lesson Notes

Working in groups, students prepare graphs of separate branches of the tangent function, combining them into a single graph. In this way, the periodicity of the tangent function becomes apparent. The slope interpretation of the tangent function is then recalled from Lesson 6 and used to develop trigonometric identities involving the tangent function. Continue to emphasize to students that a trigonometric identity consists of a statement that two functions are equal and a specification of a domain on which the statement is valid. That is, the statement "$\tan(x + \pi) = \tan(x)$" is not an identity, but the statement "$\tan(x + \pi) = \tan(x)$, for $x \neq \frac{\pi}{2} + k\pi$, for all integers k" is an identity. This lesson uses both x and θ to represent the independent variables of the tangent function; θ is used when working with a circle and x when working in the xy-plane.

Classwork

Opening (3 minutes)

In Lessons 6 and 7, the tangent function was introduced as a function of a number of degrees of rotation. Hold a quick discussion about how students are now using radians for the independent variable of the tangent function just as they do for the sine and cosine functions, and discuss how that changes the domain of the tangent function from $\{\theta \in \mathbb{R} \mid \theta \neq 90 + 180k, \text{for all integers } k\}$ to $\{\theta \in \mathbb{R} \mid \theta \neq \frac{\pi}{2} + k\pi, \text{for all integers } k\}$.

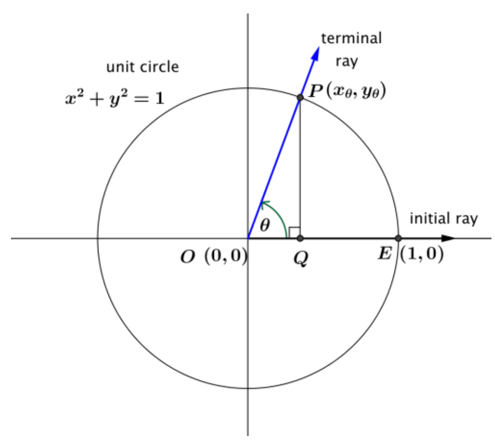
Take the opportunity to recall the working definition of the tangent function: $\tan(x) = \frac{\sin(x)}{\cos(x)}$ for $\cos(x) \neq 0$. In this lesson, students also use the slope interpretation of the tangent function from Lesson 6, so remind them that $\tan(\theta)$ is the value of the slope of the line through the terminal ray after being rotated by θ radians.

TANGENT FUNCTION: The *tangent function*,

$$\tan: \{\theta \in \mathbb{R} \mid \theta \neq \frac{\pi}{2} + k\pi, \text{for all integers } k\} \to \mathbb{R},$$

can be defined as follows: Let θ be any real number such that $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\tan(\theta)$ is $\frac{y_\theta}{x_\theta}$.

Thus, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ for all $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.



Have students express their understanding of the tangent function definition in their own words and discuss how to find a few specific values of the tangent function, for instance, $\tan\left(\frac{\pi}{4}\right)$ and $\tan\left(\frac{\pi}{3}\right)$, to ensure understanding before continuing.

Scaffolding:

- Consider demonstrating Exercise 1 for the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Alternatively, consider giving this interval to a group that may struggle.
- Ask students who are working above grade level, "How do you think these intervals were chosen?"

  Discuss other possibilities such as $[0, \pi)$, excluding $\frac{\pi}{2}$.

Exploratory Challenge 1/Exercises 1–5 (10 minutes)

Break the class into eight groups, and hand out one copy of the axes at the end of this lesson to each group. Each group is assigned one interval

$$\dots, \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right), \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \dots$$

and uses a calculator to generate approximate values of the tangent function in the tables below. Each group creates a graph of a portion of the tangent function on their set of axes using bold markers if possible. Affix each group's graph on the board so that students can see multiple branches of the tangent function at once.

Exploratory Challenge1/Exercises 1–5

1. **Use your calculator to calculate each value of $\tan(x)$ to two decimal places in the table for your group.**

Group 1 $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$		Group 2 $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$		Group 3 $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$		Group 4 $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$	
x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$
$-\frac{11\pi}{24}$	-7.60	$\frac{13\pi}{24}$	-7.60	$-\frac{35\pi}{24}$	-7.60	$\frac{37\pi}{24}$	-7.60
$-\frac{5\pi}{12}$	-3.73	$\frac{7\pi}{12}$	-3.73	$-\frac{17\pi}{12}$	-3.73	$\frac{19\pi}{12}$	-3.73
$-\frac{4\pi}{12}$	-1.73	$\frac{8\pi}{12}$	-1.73	$-\frac{16\pi}{12}$	-1.73	$\frac{20\pi}{12}$	-1.73
$-\frac{3\pi}{12}$	-1.00	$\frac{9\pi}{12}$	-1.00	$-\frac{15\pi}{12}$	-1.00	$\frac{21\pi}{12}$	-1.00
$-\frac{2\pi}{12}$	-0.58	$\frac{10\pi}{12}$	-0.58	$-\frac{14\pi}{12}$	-0.58	$\frac{22\pi}{12}$	-0.58
$-\frac{\pi}{12}$	-0.27	$\frac{11\pi}{12}$	-0.27	$-\frac{13\pi}{12}$	-0.27	$\frac{23\pi}{12}$	-0.27
0	0.00	π	0.00	$-\pi$	0.00	2π	0.00
$\frac{\pi}{12}$	0.27	$\frac{13\pi}{12}$	0.27	$-\frac{11\pi}{12}$	0.27	$\frac{25\pi}{12}$	0.27
$\frac{2\pi}{12}$	0.58	$\frac{14\pi}{12}$	0.58	$-\frac{10\pi}{12}$	0.58	$\frac{26\pi}{12}$	0.58
$\frac{3\pi}{12}$	1.00	$\frac{15\pi}{12}$	1.00	$-\frac{9\pi}{12}$	1.00	$\frac{27\pi}{12}$	1.00
$\frac{4\pi}{12}$	1.73	$\frac{16\pi}{12}$	1.73	$-\frac{8\pi}{12}$	1.73	$\frac{28\pi}{12}$	1.73
$\frac{5\pi}{12}$	3.73	$\frac{17\pi}{12}$	3.73	$-\frac{7\pi}{12}$	3.73	$\frac{29\pi}{12}$	3.73
$\frac{11\pi}{24}$	7.60	$\frac{35\pi}{24}$	7.60	$-\frac{13\pi}{24}$	7.60	$\frac{59\pi}{24}$	7.60

Group 5 $\left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right)$	
x	$\tan(x)$
$-\frac{59\pi}{24}$	-7.60
$-\frac{29\pi}{12}$	-3.73
$-\frac{28\pi}{12}$	-1.73
$-\frac{27\pi}{12}$	-1.00
$-\frac{26\pi}{12}$	-0.58
$-\frac{25\pi}{12}$	-0.27
-2π	0.00
$-\frac{23\pi}{12}$	0.27
$-\frac{22\pi}{12}$	0.58
$-\frac{21\pi}{12}$	1.00
$-\frac{20\pi}{12}$	1.73
$-\frac{19\pi}{12}$	3.73
$-\frac{37\pi}{24}$	7.60

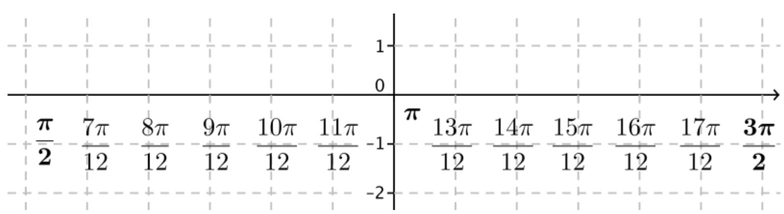
Group 6 $\left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$	
x	$\tan(x)$
$\frac{61\pi}{24}$	-7.60
$\frac{31\pi}{12}$	-3.73
$\frac{32\pi}{12}$	-1.73
$\frac{33\pi}{12}$	-1.00
$\frac{34\pi}{12}$	-0.58
$\frac{35\pi}{12}$	-0.27
3π	0.00
$\frac{37\pi}{12}$	0.27
$\frac{38\pi}{12}$	0.58
$\frac{39\pi}{12}$	1.00
$\frac{40\pi}{12}$	1.73
$\frac{41\pi}{12}$	3.73
$\frac{83\pi}{24}$	7.60

Group 7 $\left(-\frac{7\pi}{2}, -\frac{5\pi}{2}\right)$	
x	$\tan(x)$
$-\frac{83\pi}{24}$	-7.60
$-\frac{41\pi}{12}$	-3.73
$-\frac{40\pi}{12}$	-1.73
$-\frac{39\pi}{12}$	-1.00
$-\frac{38\pi}{12}$	-0.58
$-\frac{37\pi}{12}$	-0.27
-3π	0.00
$-\frac{35\pi}{12}$	0.27
$-\frac{34\pi}{12}$	0.58
$-\frac{33\pi}{12}$	1.00
$-\frac{32\pi}{12}$	1.73
$-\frac{31\pi}{12}$	3.73
$-\frac{61\pi}{24}$	7.60

Group 8 $\left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)$	
x	$\tan(x)$
$\frac{37\pi}{24}$	-7.60
$\frac{43\pi}{12}$	-3.73
$\frac{44\pi}{12}$	-1.73
$\frac{45\pi}{12}$	-1.00
$\frac{46\pi}{12}$	-0.58
$\frac{47\pi}{12}$	-0.27
4π	0.00
$\frac{49\pi}{12}$	0.27
$\frac{50\pi}{12}$	0.58
$\frac{51\pi}{12}$	1.00
$\frac{52\pi}{12}$	1.73
$\frac{53\pi}{12}$	3.73
$\frac{107\pi}{24}$	7.60

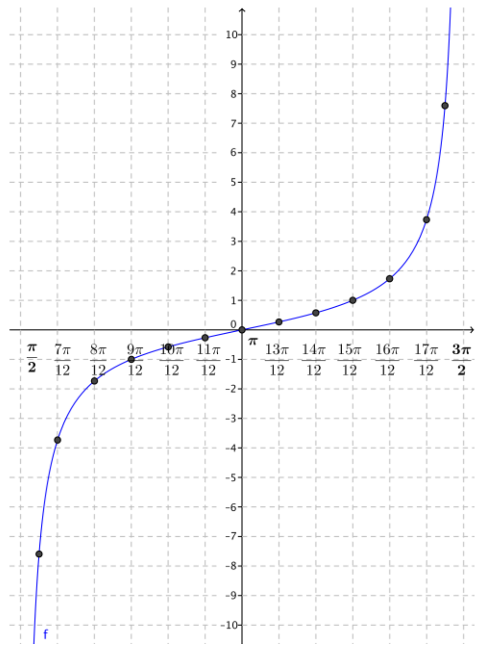
2. The tick marks on the axes provided are spaced in increments of $\frac{\pi}{12}$. Mark the horizontal axis by writing the number of the left endpoint of your interval at the leftmost tick mark, the multiple of π that is in the middle of your interval at the point where the axes cross, and the number at the right endpoint of your interval at the rightmost tick mark. Fill in the remaining values at increments of $\frac{\pi}{12}$.

The x-axis from one such set of marked axes on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is shown below.



3. **On your plot, sketch the graph of $y = \tan(x)$ on your specified interval by plotting the points in the table and connecting the points with a smooth curve. Draw the graph with a bold marker.**

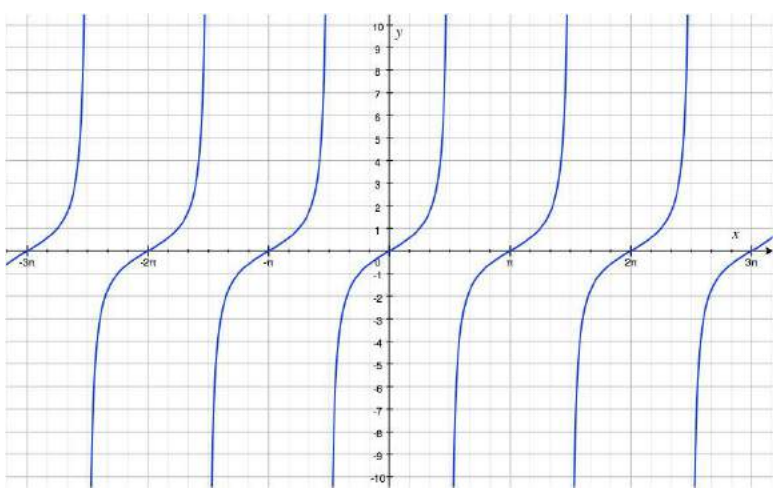
An example on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is shown to the right.



4. **What happens to the graph near the edges of your interval? Why does this happen?**

Near the edges of the interval, x approaches a value where the tangent function is undefined. Since $\cos\left(\frac{\pi}{2} + k\pi\right) = 0$ for any integer k, the tangent is undefined at the edges of the interval we are graphing. As x gets near the edge of the interval, $\sin(x)$ gets close to either 1 or -1, and $\cos(x)$ gets close to 0. Thus, $\tan(x) = \frac{\sin(x)}{\cos(x)}$ gets very far from zero, either in the positive or negative direction.

5. **When you are finished, affix your graph to the board in the appropriate place, matching endpoints of intervals.**



Discussion (2 minutes)

MP.8

- What do you notice about the graph of the tangent function?
  - *It repeats every π units (i.e., the function has period π and frequency $\frac{1}{\pi}$).*
  - *It is broken into pieces that are the same.*
  - *It breaks when $x = \frac{\pi}{2} + k\pi$, for some integer k.*
  - *Each piece has rotational symmetry by $180°$.*
  - *The graph has symmetry under horizontal translation by π units.*
  - *The entire graph has rotational symmetry by $180°$.*
- The breaks in the graph are examples of *vertical asymptotes*. These lines that the graph gets very close to but does not cross often occur at a value $x = a$, where the function is undefined to prevent division by zero.

Exploratory Challenge 1/Exercises 6–16 (15 minutes)

This is a discovery exercise for students to establish many facts or identities about the tangent function using the unit circle with the slope interpretation. The identities are verified using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ in the Problem Set and should be discussed in the context of the graph of $y = \tan(x)$ that was just made.

Scaffolding:

- For struggling students, consider working through Exercise 6 as a class before asking students to complete Exercises 7–16 in groups.
- Targeted instruction with a small group while other students work on problems independently may help struggling students.
- Consider also giving each group their choice of five problems in such a way that each problem is covered. If this is done, then tell students to be prepared to present their results and debrief the class at the end.

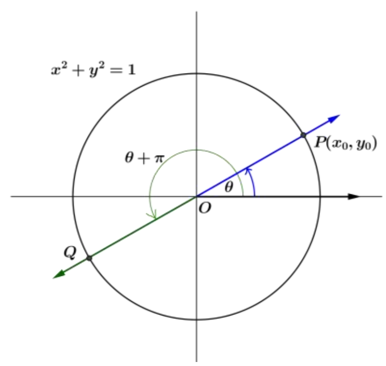
Exploratory Challenge 2/Exercises 6–16

For each exercise below, let $m = \tan(\theta)$ be the slope of the terminal ray in the definition of the tangent function, and let $P = (x_0, y_0)$ be the intersection of the terminal ray with the unit circle after being rotated by θ radians for $0 < \theta < \frac{\pi}{2}$. We know that the tangent of θ is the slope m of $\overleftrightarrow{OP}$.

6. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\theta + \pi$ radians.

a. What is the slope of $\overleftrightarrow{OQ}$?

Points P, O, and Q are collinear, so the slope of $\overleftrightarrow{OQ}$ is the same as the slope of $\overleftrightarrow{OP}$. Thus, the slope of $\overleftrightarrow{OQ}$ is also m. Another approach would be to find that the coordinates of Q are $(-x_0, -y_0)$, so the slope of ray $\overleftrightarrow{OQ}$ is $\frac{y_0}{x_0}$, which is m.



b. Find an expression for $\tan(\theta + \pi)$ in terms of m.

$\tan(\theta + \pi) = m$

c. Find an expression for $\tan(\theta + \pi)$ in terms of $\tan(\theta)$.

$\tan(\theta + \pi) = \tan(\theta)$

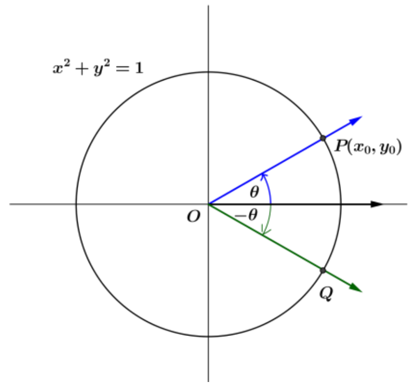
d. How can the expression in part (c) be seen in the graph of the tangent function?

The pieces of the tangent function repeat every π units because $\tan(\theta + \pi) = \tan(\theta)$.

At this time, students should have strong evidence for why the period for the tangent function is π and not 2π, but they may not be connecting the value of the tangent function from the slope interpretation to the graphs drawn at the beginning of the lesson. Continue to stress the interrelationship between the different interpretations and that information gathered from one interpretation can help students understand the function in a different interpretation.

7. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $-\theta$ radians.

a. What is the slope of $\overleftrightarrow{OQ}$?



Ray $\overrightarrow{OQ}$ is the reflection of ray $\overrightarrow{OP}$ across the x-axis, so the coordinates of Q are $(x_0, -y_0)$. Thus, the slope of $\overleftrightarrow{OQ}$ is the opposite of the slope of $\overleftrightarrow{OP}$. Thus, the slope of $\overleftrightarrow{OQ}$ is $-m$.

b. Find an expression for $\tan(-\theta)$ in terms of m.

$\tan(-\theta) = -m$

c. Find an expression for $\tan(-\theta)$ in terms of $\tan(\theta)$.

$\tan(-\theta) = -\tan(\theta)$

d. How can the expression in part (c) be seen in the graph of the tangent function?

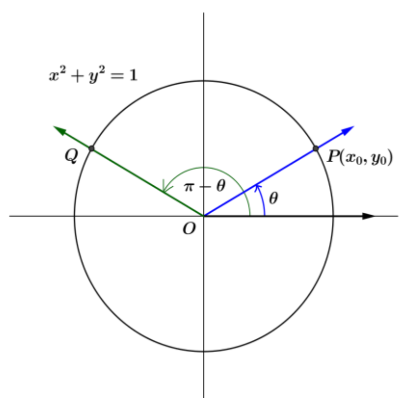
The graph of the tangent function has rotational symmetry about the origin.

8. Is the tangent function an even function, an odd function, or neither? How can you tell your answer is correct from the graph of the tangent function?

Because $\tan(-\theta) = -\tan(\theta)$, the tangent function is odd. If the graph of the tangent function is rotated π radians about the origin, there will appear to be no change in the graph.

9. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\pi - \theta$ radians.

a. What is the slope of $\overleftrightarrow{OQ}$?



Ray $\overrightarrow{OQ}$ is the reflection of ray $\overrightarrow{OP}$ across the y-axis, so the coordinates of Q are $(-x_0, y_0)$. Then, the slope of $\overleftrightarrow{OQ}$ is the opposite of the slope of $\overleftrightarrow{OP}$, so the slope of $\overleftrightarrow{OQ}$ is $-m$.

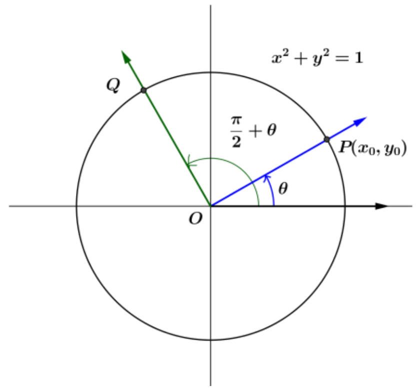
b. Find an expression for $\tan(\pi - \theta)$ in terms of m.

$\tan(\pi - \theta) = -m$

c. Find an expression for $\tan(\pi - \theta)$ in terms of $\tan(\theta)$.

$\tan(\pi - \theta) = -\tan(\theta)$

10. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\frac{\pi}{2} + \theta$ radians.



a. What is the slope of $\overleftrightarrow{OQ}$?

Ray OQ is the rotation of $\overrightarrow{OP}$ by $90°$ counterclockwise, so line OQ is perpendicular to $\overleftrightarrow{OP}$. Thus, the slopes of $\overleftrightarrow{OQ}$ and $\overleftrightarrow{OP}$ are opposite reciprocals. Thus, the slope of $\overleftrightarrow{OQ}$ is $-\frac{1}{m}$.

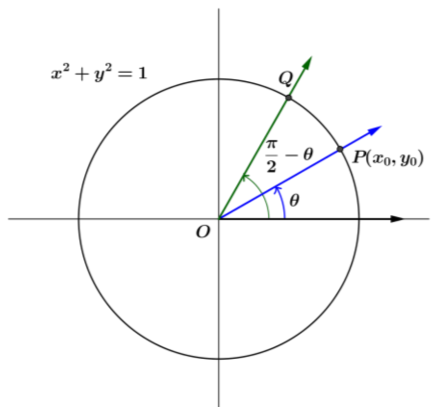
b. Find an expression for $\tan\left(\frac{\pi}{2} + \theta\right)$ in terms of m.

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{1}{m}$$

c. Find an expression for $\tan\left(\frac{\pi}{2} + \theta\right)$ first in terms of $\tan(\theta)$ and then in terms of $\cot(\theta)$.

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{1}{\tan(\theta)} = -\cot(\theta)$$

11. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\frac{\pi}{2} - \theta$ radians.



a. What is the slope of $\overleftrightarrow{OQ}$?

Ray OQI s the reflection of $\overrightarrow{OP}$ across the diagonal line $y = x$, so the coordinates of Q are (y_0, x_0). Then, the slope of $\overleftrightarrow{OQ}$ is $\frac{1}{m}$.

b. Find an expression for $\tan\left(\frac{\pi}{2} - \theta\right)$ in terms of m.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{m}$$

c. Find an expression for $\tan\left(\frac{\pi}{2} - \theta\right)$ in terms of $\tan(\theta)$ or other trigonometric functions.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

12. Summarize your results from Exercises 6, 7, 9, 10, and 11.

For $0 < \theta < \frac{\pi}{2}$:

$$\tan(\theta + \pi) = \tan(\theta);$$
$$\tan(-\theta) = -\tan(\theta);$$
$$\tan(\pi - \theta) = -\tan(\theta);$$
$$\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{1}{\tan(\theta)};$$
$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan(\theta)}.$$

13. **We have only demonstrated that the identities in Exercise 12 are valid for $0 < \theta < \frac{\pi}{2}$ because we only used rotations that left point P in the first quadrant. Argue that $\tan\left(-\frac{2\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right)$. Then, using similar logic, we could argue that all of the above identities extend to any value of θ for which the tangent (and cotangent for the last two) is defined.**

    By the property developed in Exercise 3, $\tan\left(\frac{\pi}{3}\right) = -\tan\left(\pi - \frac{\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right)$. Because the terminal ray of a rotation through $-\frac{2\pi}{3}$ radians is collinear with the terminal ray of a rotation through $\frac{\pi}{3}$ radians, $\tan\left(\frac{\pi}{3}\right) = \tan\left(-\frac{2\pi}{3}\right)$. Thus, by transitivity, we have $\tan\left(-\frac{2\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right)$.

14. **For which values of θ are the identities in Exercise 12 valid?**

    Using a process similar to the one we used in Exercise 13, we can show that the value of θ can be any real number that does not cause a zero in the denominator. The tangent function is only defined for $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k. Also, for those identities involving $\cot(\theta)$, we need to have $\theta \neq \pi k$, for all integers k.

15. **Derive an identity for $\tan(2\pi + \theta)$ from the graph.**

    Because the terminal ray for a rotation by $2\pi + \theta$ and the terminal ray for a rotation by θ coincide, we see that $\tan(2\pi + \theta) = \tan(\theta)$, where $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

16. **Use the identities you summarized in Exercise 12 to show $\tan(2\pi - \theta) = -\tan(\theta)$ where $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.**

    From Exercise 6, $\tan(2\pi - \theta) = \tan(\pi + (\pi - \theta)) = \tan(\pi - \theta) = \tan(-\theta)$.

    From Exercise 7, $\tan(-\theta) = -\tan(\theta)$.

    Thus, $\tan(2\pi - \theta) = -\tan(\theta)$ for $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

Discussion (3 minutes)

Use this opportunity to reinforce the major results of the Opening Exercise and to check for understanding of key concepts. Debrief the previous set of exercises with students, and discuss the identities for the tangent function derived in Exercises 6, 7, 9, 10, 11, 15, and 16. Be sure that every student has the correct identities recorded before moving on.

- What is the period of the tangent function?
    - π
- What is the value of the tangent function at $\theta + \pi$? For which values of θ is this an identity?
    - $\tan(\theta + \pi) = \tan(\theta)$ for $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.
- What is the value of the tangent function at $\pi - \theta$? For which values of θ is this an identity?
    - $\tan(\pi - \theta) = -\tan(\theta)$ for $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.
- What is the value of the tangent function at $\frac{\pi}{2} - \theta$? For which values of θ is this an identity?
    - $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$ for $\theta \neq \frac{\pi}{2} + k\pi$ and $\theta \neq k\pi$, for all integers k.

- What is the value of the tangent function at $2\pi - \theta$? For which values of θ is this an identity?
  - $\tan(2\pi - \theta) = -\tan(\theta)$ for $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

Example (4 minutes)

Part of the Problem Set for this lesson includes analytically justifying some of the tangent identities developed geometrically in the exercises. Lead the class through the following discussion to demonstrate similarities between some identities of the sine, cosine, and tangent functions.

- Compare these three identities.

$$\sin(\theta + 2\pi) = \sin(\theta), \text{for all real numbers } \theta$$
$$\cos(\theta + 2\pi) = \cos(\theta), \text{ for all real numbers } \theta$$
$$\tan(\theta + \pi) = \tan(\theta), \text{for } \theta \neq \frac{\pi}{2} + k\pi, \text{for all integers } k$$

  - *These identities come from the fundamental period of each of the three main trigonometric functions. The period of sine and cosine are 2π, and the period of tangent is π. These identities demonstrate that if you rotate the initial ray through θ radians, the values of the sine and cosine are the same whether or not you rotate through an additional 2π radians. For tangent, the value of the tangent function is the same whether you rotate the initial ray through θ radians or rotate through an additional half-turn of π radians.*

- Compare these three identities.

$$\sin(-\theta) = -\sin(\theta), \text{for all real numbers } \theta$$
$$\cos(-\theta) = \cos(\theta), \text{ for all real numbers } \theta$$
$$\tan(-\theta) = -\tan(\theta), \text{ for } \theta \neq \frac{\pi}{2} + k\pi, \text{for all integers } k$$

  - *Both the sine and tangent functions are odd functions, while the cosine function is an even function. The first two identities yield the third because* $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$.

Consider presenting other identities and having the class compare and contrast them as time permits. Another possibility would be the following identities.

$$\sin(\theta + \pi) = -\sin(\theta), \text{for all real numbers } \theta$$
$$\cos(\theta + \pi) = -\cos(\theta), \text{ for all real numbers } \theta$$
$$\tan(\theta + \pi) = \tan(\theta), \text{ for } \theta \neq \frac{\pi}{2} + k\pi, \text{for all integers } k$$

Closing (3 minutes)

Have students summarize the new identities they learned for the tangent function, both in words and symbolic notation, with a partner or in writing. Have students draw a graph of $y = \tan(x)$, including at least two full periods. Use this as an opportunity to check for any gaps in understanding.

Lesson Summary

The tangent function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is periodic with period π. The following identities have been established.

- **$\tan(x + \pi) = \tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.**
- **$\tan(-x) = -\tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.**
- **$\tan(\pi - x) = -\tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.**
- **$\tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$ for all $x \neq k\pi$, for all integers k.**
- **$\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$ for all $x \neq k\pi$, for all integers k.**
- **$\tan(2\pi + x) = \tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.**
- **$\tan(2\pi - x) = -\tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 14: Graphing the Tangent Function

Exit Ticket

1. Sketch a graph of the function $f(x) = \tan(x)$, marking the important features of the graph.

2. Given $\tan(x) = 7$, find the following function values:
   a. $\tan(\pi + x)$

   b. $\tan(2\pi - x)$

   c. $\tan\left(\frac{\pi}{2} + x\right)$

Exit Ticket Sample Solutions

1. **Sketch a graph of the function $f(x) = \tan(x)$, marking the important features of the graph.**

   Students should mark the period, the $180°$ rotational symmetry, and the vertical asymptotes where the function is undefined.

2. **Given $\tan(x) = 7$, find the following function values:**

   a. $\tan(\pi + x)$

      $\tan(\pi + x) = \tan(x) = 7$

   b. $\tan(2\pi - x)$

      $\tan(2\pi - x) = \tan(\pi - x) = -\tan(x) = -7$

   c. $\tan\left(\frac{\pi}{2} + x\right)$

      $\tan\left(\frac{\pi}{2} + x\right) = -\cot(x) = -\frac{1}{\tan(x)} = -\frac{1}{7}$

Problem Set Sample Solutions

In the first problem in this Problem Set, students construct the graph of the cotangent function. In Lesson 10, students considered the general form of a sinusoid function $f(x) = A\sin\big(\omega(x-h)\big) + k$, and in the second problem, students consider functions of the form $f(x) = A\tan\big(\omega(x-h)\big) + k$. As before, they study how the parameters A, ω, h and k affect the shape of the graph, leading to finding parameters to align the graphs of a transformed tangent function and the cotangent function.

1. **Recall that the cotangent function is defined by $\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$, where $\sin(x) \neq 0$.**

   a. **What is the domain of the cotangent function? Explain how you know.**

      Since the cotangent function is given by $\cot(x) = \frac{\cos(x)}{\sin(x)}$, the cotangent function is undefined at values of x, where $\sin(x) = 0$. This happens at values of θ that are multiples of π.

   b. **What is the period of the cotangent function? Explain how you know.**

      Since the cotangent function is the reciprocal of the tangent function, they will have the same period, π. That is, $\cot(x + \pi) = \frac{1}{\tan(x+\pi)} = \frac{1}{\tan(x)} = \cot(x)$.

c. **Use a calculator to complete the table of values of the cotangent function on the interval $(0, \pi)$ to two decimal places.**

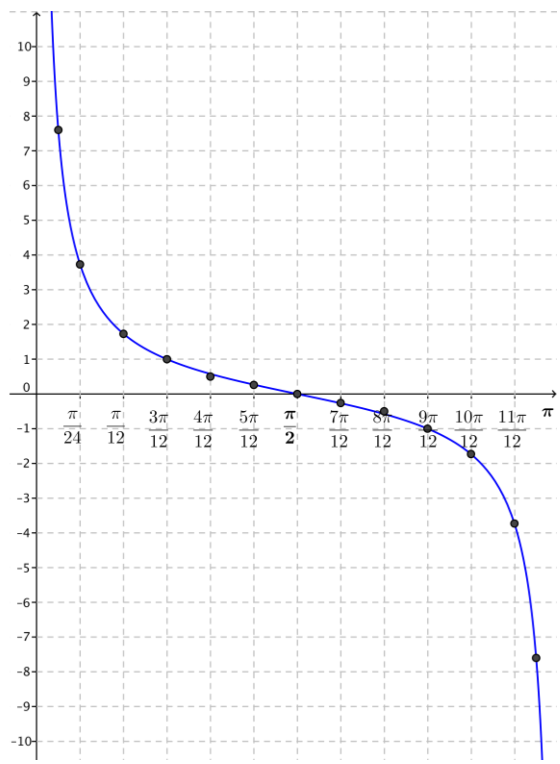
x	$\cot(x)$
$\frac{\pi}{24}$	7.60
$\frac{\pi}{12}$	3.73
$\frac{2\pi}{12}$	1.73
$\frac{3\pi}{12}$	1.00

x	$\cot(x)$
$\frac{4\pi}{12}$	0.50
$\frac{5\pi}{12}$	0.27
$\frac{\pi}{2}$	0.00

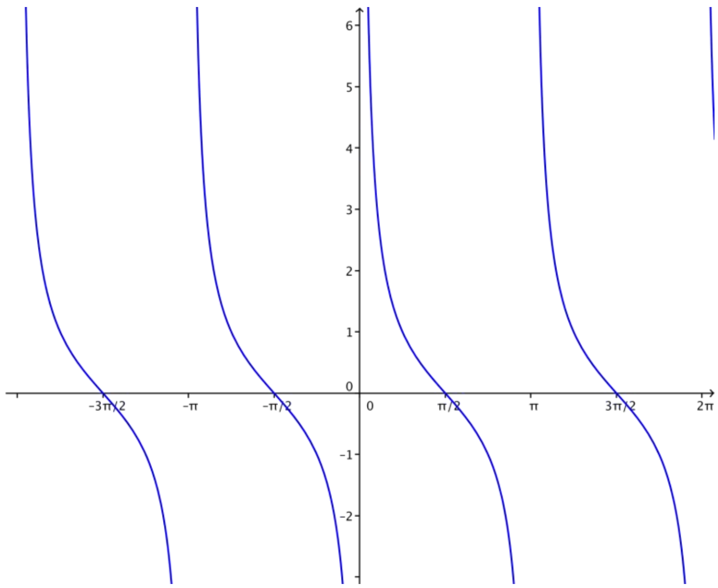
x	$\cot(x)$
$\frac{7\pi}{12}$	-0.27
$\frac{8\pi}{12}$	-0.50
$\frac{9\pi}{12}$	-1.00

x	$\cot(x)$
$\frac{10\pi}{12}$	-1.73
$\frac{11\pi}{12}$	-3.73
$\frac{23\pi}{24}$	-7.60

d. **Plot your data from part (c), and sketch a graph of $y = \cot(x)$ on $(0, \pi)$.**



e. **Sketch a graph of $y = \cot(x)$ on $(-2\pi, 2\pi)$ without plotting points.**



f. **Discuss the similarities and differences between the graphs of the tangent and cotangent functions.**

Both the tangent and cotangent functions have vertical asymptotes; the cotangent graph has vertical asymptotes at $x = k\pi$, for integers k, and the tangent has vertical asymptotes at $x = \frac{\pi}{2} + k\pi$, for integers k. The values of the tangent function increase as we look at each piece of the graph from left to right, and the values of the cotangent function decrease as we look at each piece from left to right.

g. **Find all x-values where $\tan(x) = \cot(x)$ on the interval $(0, 2\pi)$.**

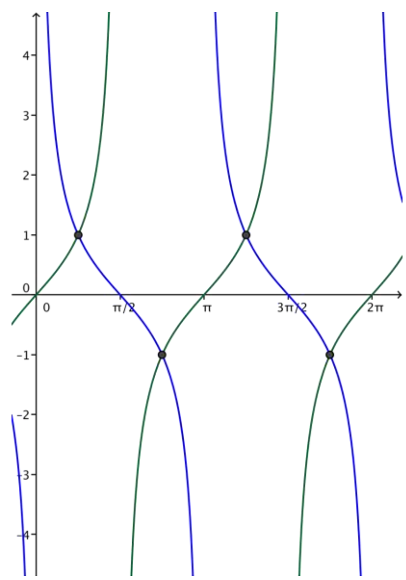
Suppose that $\tan(x) = \cot(x)$. Then, $\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\sin(x)}$.

Cross multiplying gives

$$\left(\sin(x)\right)^2 = \left(\cos(x)\right)^2$$
$$\left(\sin(x)\right)^2 - \left(\cos(x)\right)^2 = 0$$
$$\left(\sin(x) - \cos(x)\right)\left(\sin(x) + \cos(x)\right) = 0$$

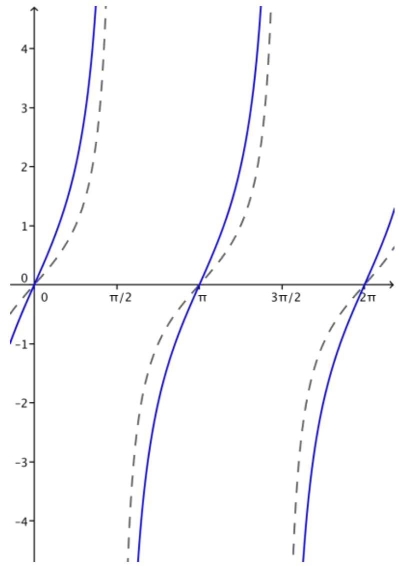
$\sin(x) = \cos(x)$ ***or*** $\sin(x) = -\cos(x)$.

The solutions are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. These are the x-values of the points of intersection of the graphs of the two functions shown to the right.

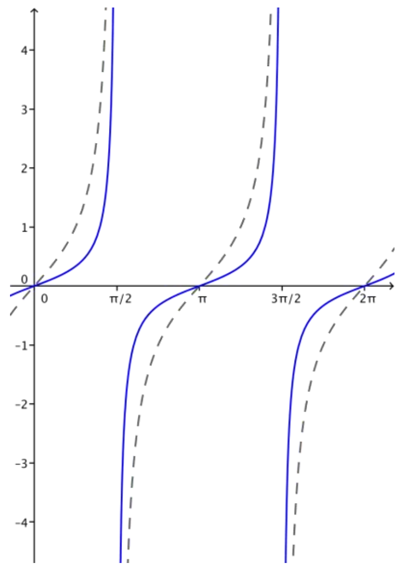


2. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

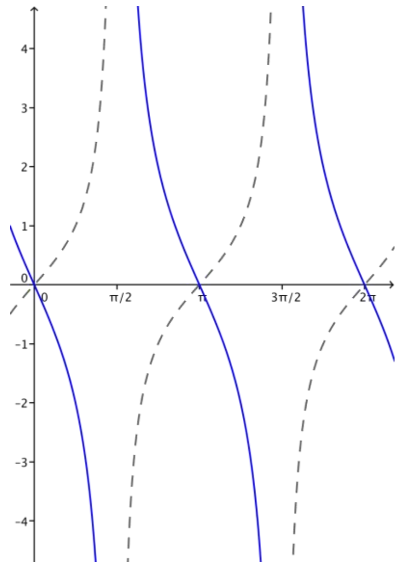
a. $g(x) = 2\tan(x)$



b. $g(x) = \frac{1}{3}\tan(x)$



c. $g(x) = -2\tan(x)$

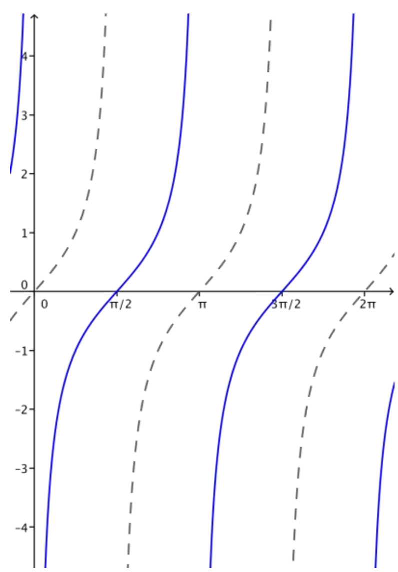


d. How does changing the parameter A affect the graph of $g(x) = A\tan(x)$?

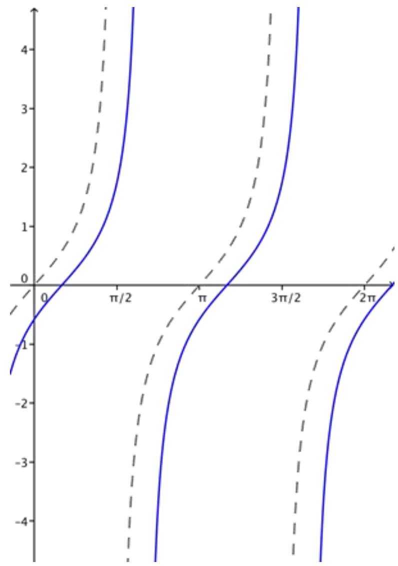
If A is positive, the graph is scaled vertically by a factor of A. If A is negative, then the graph is reflected over the x-axis and scaled vertically by a factor of $|A|$.

3. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

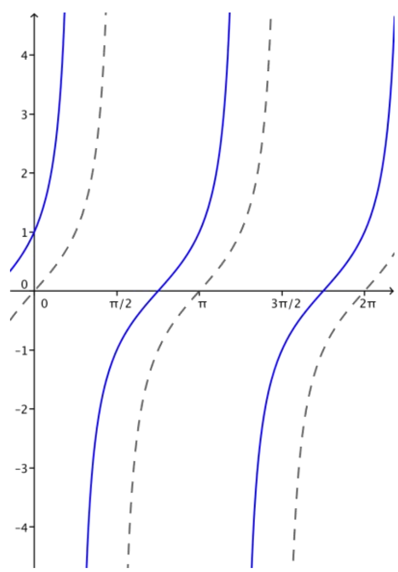
a. $g(x) = \tan\left(x - \frac{\pi}{2}\right)$



b. $g(x) = \tan\left(x - \frac{\pi}{6}\right)$



c. $g(x) = \tan\left(x + \frac{\pi}{4}\right)$

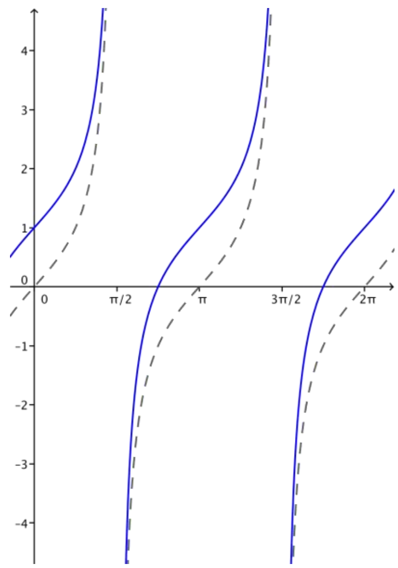


d. How does changing the parameter h affect the graph of $g(x) = \tan(x - h)$?

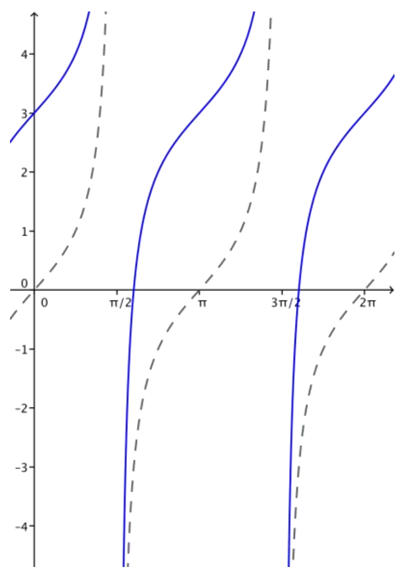
If $h > 0$, the graph is translated horizontally to the right by h units, and if $h < 0$, the graph is translated horizontally to the left by h units.

4. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

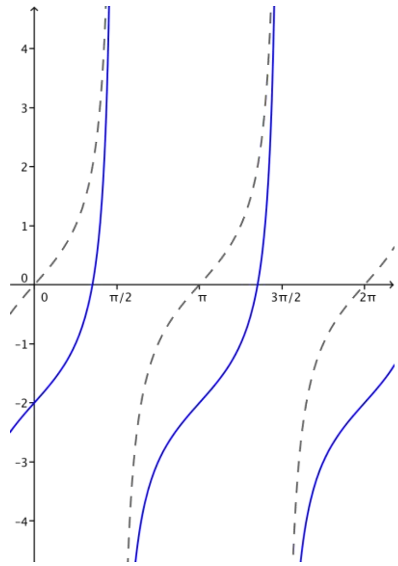
a. $g(x) = \tan(x) + 1$



b. $g(x) = \tan(x) + 3$



c. $g(x) = \tan(x) - 2$

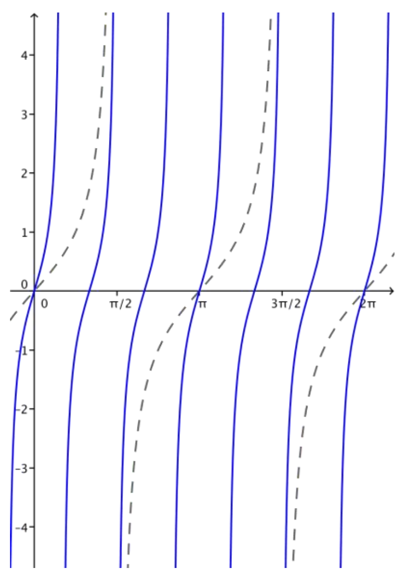


d. How does changing the parameter k affect the graph of $g(x) = \tan(x) + k$?

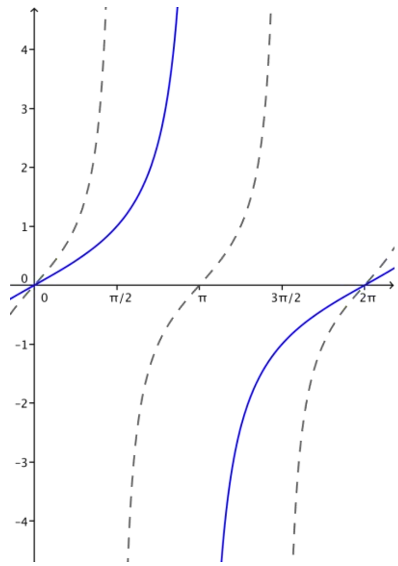
If $k > 0$, the graph is translated vertically upward by k units, and if $k < 0$, the graph is translated vertically downward by k units.

5. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

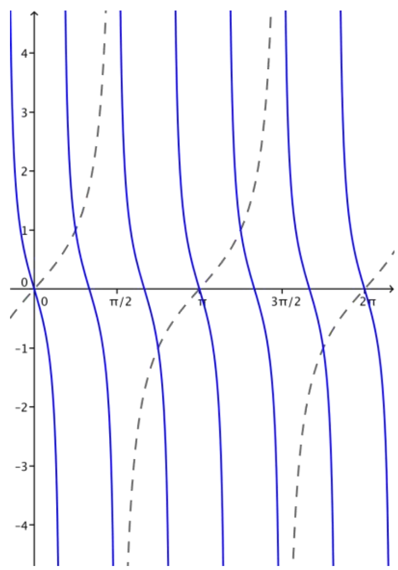
a. $g(x) = \tan(3x)$



b. $g(x) = \tan\left(\frac{x}{2}\right)$



c. $g(x) = \tan(-3x)$

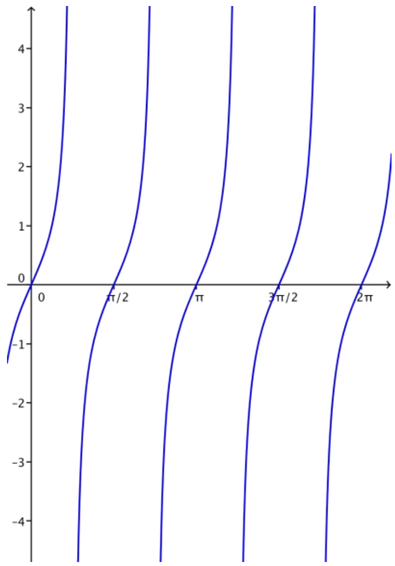


d. **How does changing the parameter ω affect the graph of $g(x) = \tan(\omega x)$?**

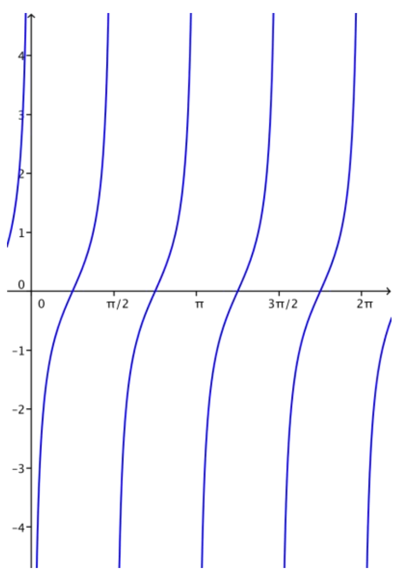
Changing ω changes the period of the graph. The period is $P = \frac{\pi}{|\omega|}$.

6. Use your knowledge of function transformation and the graph of $y = \tan(x)$ to sketch graphs of the following transformations of the tangent function.

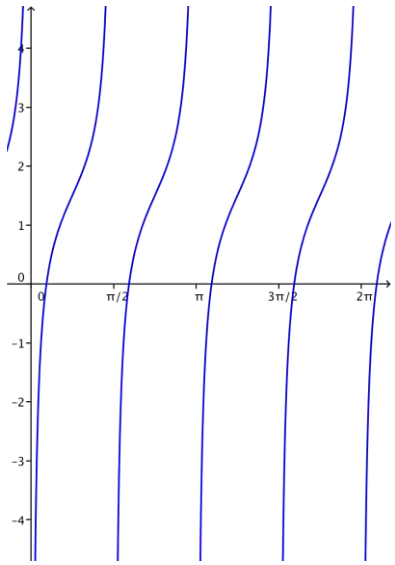
   a. $y = \tan(2x)$



   b. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$



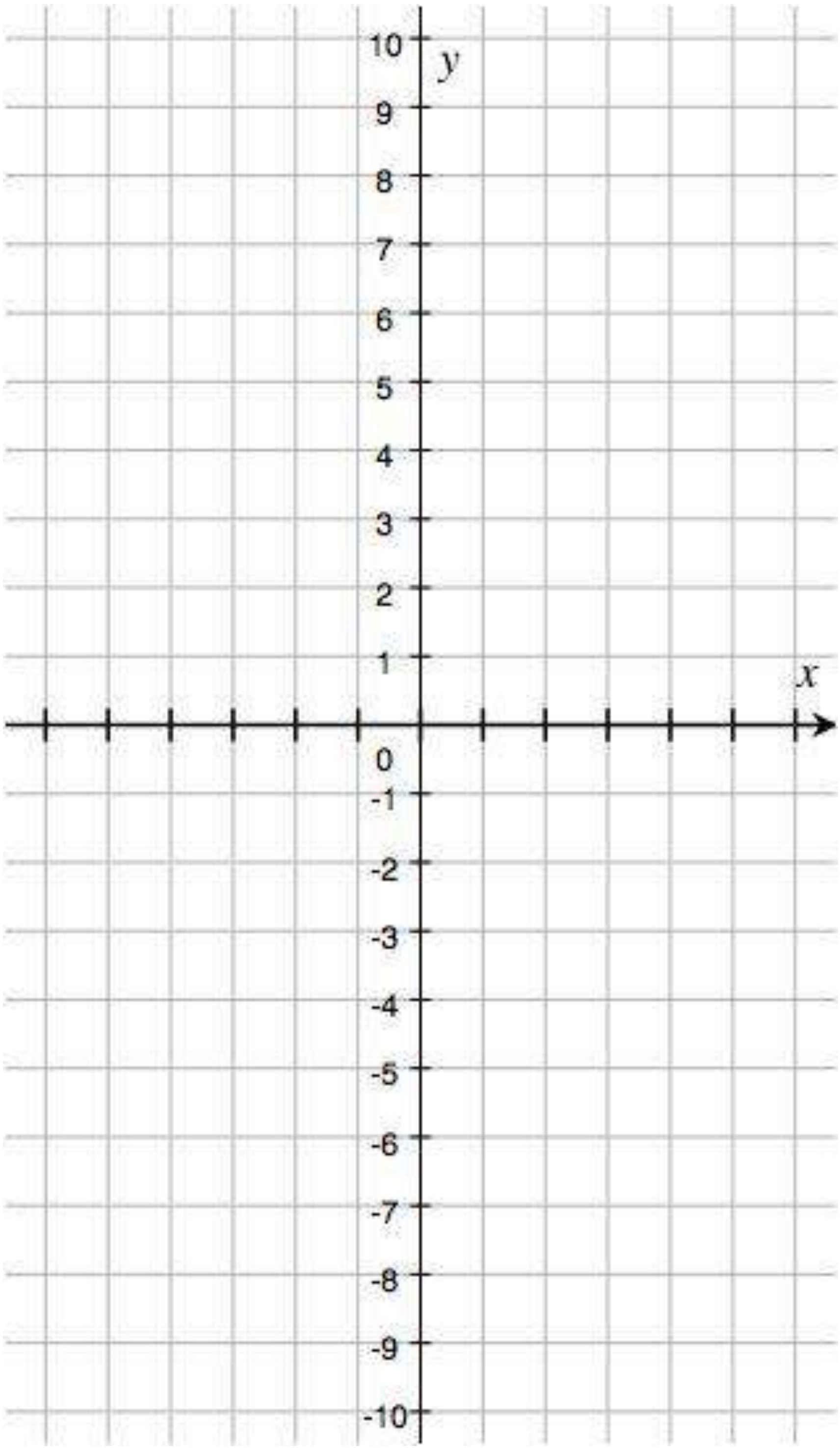
c. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right) + 1.5$



7. Find parameters A, ω, h, and k so that the graphs of $f(x) = A\tan(\omega(x - h)) + k$ and $g(x) = \cot(x)$ are the same.

The graphs of $f(x) = -\tan\left(x - \frac{\pi}{2}\right)$ and $g(x) = \cot(x)$ are the same graph.

Exploratory Challenge: Axes for Graph of Tangent Function



Lesson 15: What Is a Trigonometric Identity?

Student Outcomes

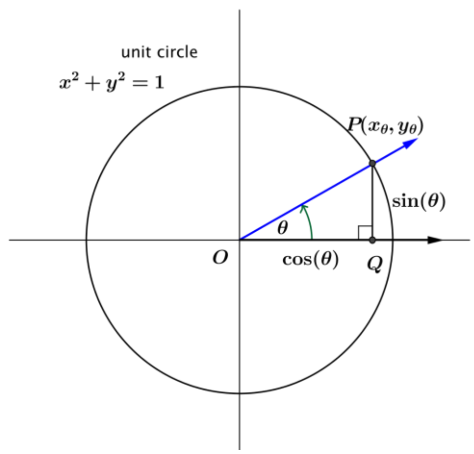
- Students prove the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$.
- Students extend trigonometric identities to the real line, with attention to domain and range.
- Students use the Pythagorean identity to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$, given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the terminal ray of the rotation.

Lesson Notes

The lesson begins with an example that develops and proves the Pythagorean identity for all real numbers. An equivalent form of the Pythagorean identity is developed, and students observe that there are special values for which the resulting functions are not defined; therefore, there are values for which the identity does not hold. Students then examine the domains for several identities and use the Pythagorean identity to find one function in terms of another in a given quadrant.

Classwork

Opening (5 minutes): The Pythagorean Identity



Lessons 4 and 5 extended the definitions of the sine and cosine functions so that $\sin(\theta)$ and $\cos(\theta)$ are defined for all real numbers θ.

- What is the equation of the unit circle centered at the origin?
  - *The unit circle has equation $x^2 + y^2 = 1$.*
- Recall that this equation is a special case of the Pythagorean theorem. Given the number θ, there is a unique point P on the unit circle that results from rotating the positive x-axis through θ radians around the origin. What are the coordinates of P?
  - *The coordinates of P are (x_θ, y_θ), where $x_\theta = \cos(\theta)$, and $y_\theta = \sin(\theta)$.*
- How can you combine this information to get a formula involving $\sin(\theta)$ and $\cos(\theta)$?
  - *Replace x and y in the equation of the unit circle by the coordinates of P. That gives $\sin^2(\theta) + \cos^2(\theta) = 1$, where θ is any real number.*
- Notice that we use the notation $\sin^2(\theta)$ in place of $(\sin(\theta))^2$. Both are correct, but the first is notationally simpler. Notice also that neither is the same expression as $\sin(\theta^2)$.

> *Scaffolding:*
> - English language learners may need support such as choral recitation or a graphic organizer for learning the word *identity*.
> - An alternative, more challenging Opening might be, "Prove that $\sin^2(\theta) + \cos^2(\theta) = 1$, for all real values of θ." (MP.3)

- The equation $\sin^2(\theta) + \cos^2(\theta) = 1$ is true for all real numbers θ and is an *identity*. The functions on either side of the equal sign are equivalent for every value of θ. They have the same domain, the same range, and the same rule of assignment. You saw some polynomial identities in Module 1, and we've developed some identities for sine, cosine, and tangent observed from graphs in this module. The identity we just proved is a trigonometric identity, and it is called the *Pythagorean identity* because it is another important consequence of the Pythagorean theorem.

Example 1 (8 minutes): Another Identity?

This example gets into the issue of what an identity is. Students should work in pairs to answer the questions.

> *Scaffolding:*
> - Circulate to identify student pairs who might be prepared to share their results and to assist any students having trouble.
> - Model the substitution process, if needed, showing explicitly how the equation becomes $\tan^2(\theta) + 1 = \frac{1}{\cos^2(\theta)}$.

- Divide both sides of the Pythagorean identity by $\cos^2(\theta)$. What happens to the identity?
  - *The equation becomes* $\tan^2(\theta) + 1 = \frac{1}{\cos^2(\theta)}$, which we can restate as $\tan^2(\theta) + 1 = \sec^2(\theta)$. *This appears to be another identity.*
- What happens when $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$, and $\theta = \frac{3\pi}{2}$? Why?
  - *Both* $\tan(\theta)$ *and* $\sec(\theta)$ *are undefined at these values of* θ. *That happens because* $\cos(\theta) = 0$ *for those values of* θ, *and we cannot divide by zero. Therefore, we can no longer say that the equation is true for all real numbers* θ.
- How do we need to modify our claim about what looks like a new identity?
  - *We need to say that* $\tan^2(\theta) + 1 = \sec^2(\theta)$ *is a trigonometric identity for all real numbers* θ *such that the functions* $f(\theta) = \tan^2(\theta) + 1$ *and* $g(\theta) = \sec^2(\theta)$ *are defined. In some cases, the functions are defined, and in other cases, they are not defined.*
- For which values of θ are the functions $f(\theta) = \tan^2(\theta) + 1$ and $g(\theta) = \sec^2(\theta)$ defined?
  - *The function* $f(\theta) = \tan^2(\theta) + 1$ *is defined for* $\theta \neq \frac{\pi}{2} + k\pi$, *for all integers* k.
  - *The function* $f(\theta) = \sec^2(\theta)$ *is defined for* $\theta \neq \frac{\pi}{2} + k\pi$, *for all integers* k.
- What is the range of each of the functions $f(\theta) = \tan^2(\theta) + 1$ and $g(\theta) = \sec^2(\theta)$?
  - *The ranges of the functions are all real numbers* $f(\theta) \geq 1$ *and* $g(\theta) \geq 1$.
- The two functions have the same domain and the same range, and they are equivalent.
- Therefore, $\tan^2(\theta) + 1 = \sec^2(\theta)$ is a trigonometric identity for all real numbers θ such that $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k. For any trigonometric identity, we need to specify not only the two functions that are equivalent, but also the values for which the identity is true.

Be sure that the discussion clarifies that any equation is not automatically an identity. The equation needs to involve the equivalence of two functions and include the specification of their identical domains.

Exercises 1–3 (25 minutes)

Students should work on these exercises individually and then share their results either in a group or with the whole class. If some students are struggling, they should be encouraged to work together with the teacher while the others work individually.

Exercises 1–3

1. **Recall the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, where θ is any real number.**

   a. **Find $\sin(x)$, given $\cos(x) = \frac{3}{5}$, for $-\frac{\pi}{2} < x < 0$.**

   From the Pythagorean identity, $\sin^2(x) + \left(\frac{3}{5}\right)^2 = 1$. So, $\sin^2(x) = 1 - \frac{9}{25} = \frac{16}{25}$, and $\sin(x) = -\frac{4}{5}$, or $\sin(x) = \frac{4}{5}$. Since $-\frac{\pi}{2} < x < 0$, we know that $\sin(x)$ is negative; therefore, $\sin(x) = -\frac{4}{5}$.

   b. **Find $\tan(y)$, given $\cos(y) = -\frac{5}{13}$, for $\frac{\pi}{2} < y < \pi$.**

   From the Pythagorean identity, $\sin^2(y) + \left(-\frac{5}{13}\right)^2 = 1$. So, $\sin^2(y) = 1 - \frac{25}{169} = \frac{144}{169}$, and $\sin(y) = -\frac{12}{13}$, or $\sin(y) = \frac{12}{13}$. Since $\frac{\pi}{2} < y < \pi$, we know that $\sin(y)$ is positive; therefore, $\sin(y) = \frac{12}{13}$.

   Therefore, $\tan(y) = \frac{\sin(y)}{\cos(y)} = \frac{\frac{12}{13}}{-\frac{5}{13}} = -\frac{12}{5}$.

   c. **Write $\tan(z)$ in terms of $\cos(z)$, for $\pi < z < \frac{3\pi}{2}$.**

   From the Pythagorean identity, $\sin^2(z) = 1 - \cos^2(z)$. Therefore, either $\sin(z) = -\sqrt{1 - \cos^2(z)}$, or $\sin(z) = \sqrt{1 - \cos^2(z)}$. Since $\sin(z)$ is negative for $\pi < x < \frac{3\pi}{2}$, $\sin(z) = -\sqrt{1 - \cos^2(z)}$. Because $\tan(z) = \frac{\sin(z)}{\cos(z)}$, we have $\tan(z) = \frac{-\sqrt{1-\cos^2(z)}}{\cos(z)}$.

2. **Use the Pythagorean identity to do the following:**

   a. **Rewrite the expression $\cos(\theta)\sin^2(\theta) - \cos(\theta)$ in terms of a single trigonometric function. State the resulting identity.**

   $$\begin{aligned}\cos(\theta)\sin^2(\theta) - \cos(\theta) &= \cos(\theta)(\sin^2(\theta) - 1)\\ &= \cos(\theta)(-\cos^2(\theta))\\ &= -\cos^3(\theta)\end{aligned}$$

   Therefore, $\cos(\theta)\sin^2(\theta) - \cos(\theta) = -\cos^3(\theta)$ for all real numbers θ.

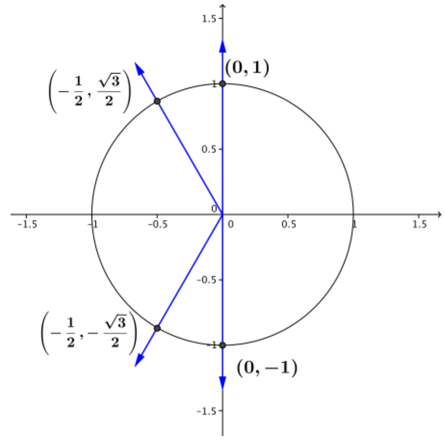
   b. **Rewrite the expression $(1 - \cos^2(\theta))\csc(\theta)$ in terms of a single trigonometric function. State the resulting identity.**

   $$\begin{aligned}(1 - \cos^2(\theta))\csc(\theta) &= \sin^2(\theta)\csc(\theta)\\ &= \sin^2(\theta)\frac{1}{\sin(\theta)}\\ &= \sin(\theta)\end{aligned}$$

   Therefore, $(1 - \cos^2(\theta))\csc(\theta) = \sin(\theta)$ for $\theta \neq k\pi$, for all integers k.

c. Find all solutions to the equation $2\sin^2(\theta) = 2 + \cos(\theta)$ in the interval $(0, 2\pi)$. Draw a unit circle that shows the solutions.

$$2\sin^2(\theta) = 2 + \cos(\theta)$$
$$2(1 - \cos^2(\theta)) = 2 + \cos(\theta)$$
$$2 - 2\cos^2(\theta) = 2 + \cos\theta$$
$$-2\cos^2(\theta) - \cos(\theta) = 0$$
$$2\cos^2(\theta) + \cos(\theta) = 0$$
$$\cos(\theta)\,(2\cos(\theta) + 1) = 0$$



Therefore, $\cos(\theta) = 0$*, or* $\cos(\theta) = -\frac{1}{2}$*.*

See the unit circle on the right, which shows the four points where $\cos(\theta) = 0$*, or* $\cos(\theta) = -\frac{1}{2}$*.*

In the interval $(0, 2\pi)$*,* $\cos(\theta) = 0$ *only if* $\theta = \frac{\pi}{2}$*, or* $\theta = \frac{3\pi}{2}$*. Also,* $\cos(\theta) = -\frac{1}{2}$ *only if* $\theta = \frac{2\pi}{3}$*, or* $\theta = \frac{4\pi}{3}$*.*
Therefore, the solutions of the equations in the interval $(0, 2\pi)$ *are* $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}$*, and* $\frac{4\pi}{3}$*.*

3. Which of the following statements are identities? If a statement is an identity, specify the values of x where the equation holds.

a. $\sin(x + 2\pi) = \sin(x)$ where the functions on both sides are defined.

This is an identity defined for all real numbers.

b. $\sec(x) = 1$ where the functions on both sides are defined.

This is not an identity. The functions are not equivalent for all real numbers. For example, although $\sec(0) = 1$*,* $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$*. The functions have equal values only when* x *is an integer multiple of* 2π*. Additionally, the ranges are different. The range of* $f(x) = \sec(x)$ *is all real numbers* y *such that* $y \le -1$ *or* $y \ge 1$*, whereas the range of* $g(x) = 1$ *is the single number* 1*.*

c. $\sin(-x) = \sin(x)$ where the functions on both sides are defined.

This is not an identity; this statement is only true when $\sin(x) = 0$*, which happens only at integer multiples of* π*.*

d. $1 + \tan^2(x) = \sec^2(x)$ where the functions on both sides are defined.

This is an identity. The functions on either side are defined for $\theta \ne \frac{\pi}{2} + k\pi$*, for all integers* k*.*

e. $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$ where the functions on both sides are defined.

This is an identity defined for all real numbers.

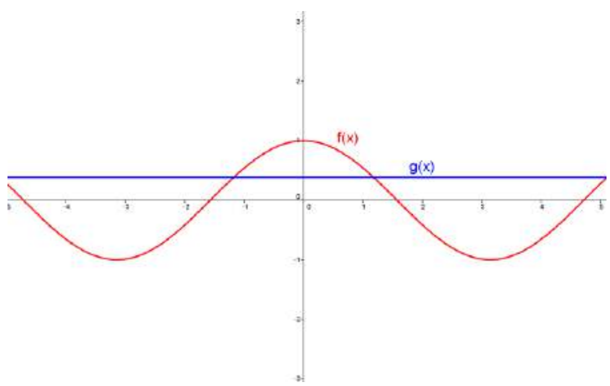
f. $\sin^2(x) = \tan^2(x)$ for all real x.

This is not an identity. The equation $\sin^2(x) = \tan^2(x)$ is only true where $\sin^2(x) = \frac{\sin^2(x)}{\cos^2(x)}$, so $\cos^2(x) = 1$, and then $\cos(x) = 1$, or $\cos(x) = -1$, which gives $x = \pi k$, for all integers k. For all other values of x, the functions on the two sides are not equal. Moreover, $\tan^2(x)$ is defined only for $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k, whereas $\sin^2(x)$ is defined for all real numbers.

Another argument for why this statement is not an identity is that $\sin^2\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$, but $\tan^2\left(\frac{\pi}{4}\right) = 1^2 = 1$, and $1 \neq \frac{1}{2}$; therefore, the statement is not true for all values of x.

Closing (2 minutes)

- One trigonometric equation is $\cos(x) = \frac{5}{13}$. Explain why this equation is *not* a trigonometric identity.
  - *The functions on each side of the equal sign have the same domain. The left side, $f(x) = \cos(x)$, is defined for all real x. The right side, $g(x) = \frac{5}{13}$, is also defined for all real x.*
  - *The two functions are not, however, equivalent. The left side is a trigonometric function that equals $\frac{5}{13}$ only sometimes. For example, if $x = 0$, then $\cos(x) = 1$. The right side, in contrast, is a constant function. The functions $f(x)$ and $g(x)$ are not equal on all values for which they are defined.*
  - *The range of $f(x) = \cos(x)$ is the set of all real numbers y such that $-1 \leq y \leq 1$, whereas the range of $g(x) = \frac{5}{13}$ is the single number $y = \frac{5}{13}$. This is further evidence that the two functions are different.*
  - *The graphs of the two functions show how the functions are different. The graph of $f(x) = \cos(x)$ is periodic, whereas the graph of $g(x) = \frac{5}{13}$ is a horizontal line. See the graphs below.*



  - *Because the two functions are not equivalent wherever they are defined, the equation is not an identity.*

Lesson Summary

The Pythagorean identity: $\sin^2(\theta) + \cos^2(\theta) = 1$ for all real numbers θ.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 15: What Is a Trigonometric Identity?

Exit Ticket

April claims that $1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$ is an identity for all real numbers θ that follows from the Pythagorean identity.

a. For which values of θ are the two functions $f(\theta) = 1 + \frac{\cos^2(\theta)}{\sin^2(\theta)}$ and $g(\theta) = \frac{1}{\sin^2(\theta)}$ defined?

b. Show that the equation $1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$ follows from the Pythagorean identity.

c. Is April correct? Explain why or why not.

d. Write the equation $1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$ in terms of other trigonometric functions, and state the resulting identity.

Exit Ticket Sample Solutions

April claims that $1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$ is an identity for all real numbers θ that follows from the Pythagorean identity.

a. For which values of θ are the two functions $f(\theta) = 1 + \frac{\cos^2(\theta)}{\sin^2(\theta)}$ and $g(\theta) = \frac{1}{\sin^2(\theta)}$ defined?

Both functions contain $\sin(\theta)$ in the denominator, so they are undefined if $\sin(\theta) = 0$. Thus, the two functions f and g are defined when $\theta \neq k\pi$, for all integers k.

b. Show that the equation $1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$ follows from the Pythagorean identity.

By the Pythagorean identity, $\sin^2(\theta) + \cos^2(\theta) = 1$.

If $\sin(\theta) \neq 0$, then,

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$
$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}.$$

c. Is April correct? Explain why or why not.

No. While April's equation does follow from the Pythagorean identity, it is not valid for all real numbers θ. For example, if $\theta = \pi$, then both sides of the equation are undefined. In order to divide by $\sin^2(\theta)$, we need to be sure that we are not dividing by zero.

d. Write the equation $1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$ in terms of other trigonometric functions, and state the resulting identity.

Because $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ and $\csc(\theta) = \frac{1}{\sin(\theta)}$, we can rewrite the equation as

$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$
$$1 + \left(\frac{\cos(\theta)}{\sin(\theta)}\right)^2 = \left(\frac{1}{\sin(\theta)}\right)^2$$
$$1 + \cot^2(\theta) = \csc^2(\theta).$$

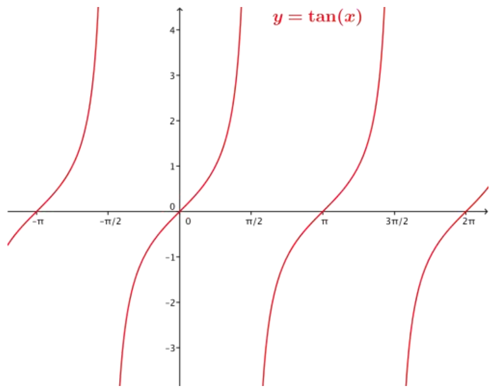
Thus, $1 + \cot^2(\theta) = \csc^2(\theta)$, where $\theta \neq k\pi$, for all integers k.

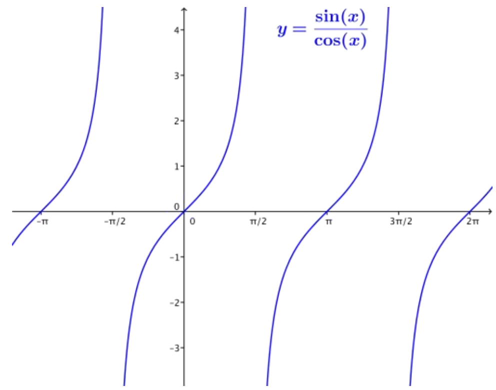
Problem Set Sample Solutions

Problems are intended to give students practice in distinguishing trigonometric identities from other trigonometric equations, in distinguishing identities defined for all real numbers from those that are defined on a subset of the real numbers, and in using the Pythagorean identity and given information to find values of trigonometric functions.

1. Which of the following statements are trigonometric identities? Graph the functions on each side of the equation.

a. $\tan(x) = \frac{\sin(x)}{\cos(x)}$ **where the functions on both sides are defined.**

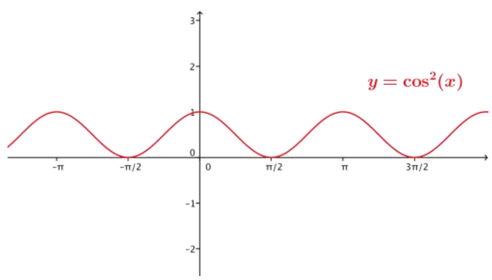


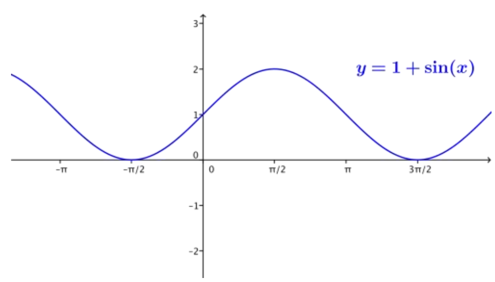


This is an identity that is defined for $x \neq \frac{\pi}{2} + k\pi$, for all integers k. See the identical graphs above.

b. $\cos^2(x) = 1 + \sin(x)$ **where the functions on both sides are defined.**

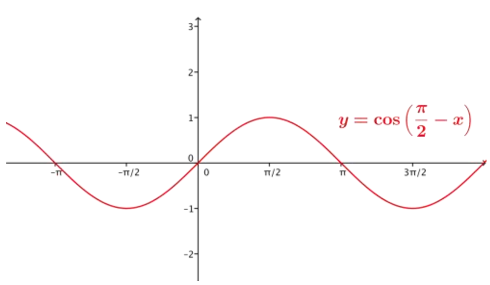
This is not an identity. For example, when $x = 0$, the left side of the equation is 1, and the right side is also 1. But when $x = \frac{\pi}{2}$, the left side is 0, and the right side is 2. The graphs below are clearly different.

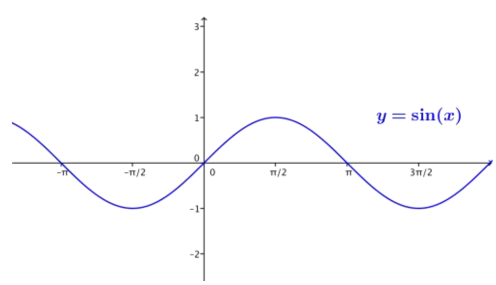




c. $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ **where the functions on both sides are defined.**

This is an identity that is defined for all real numbers x. See the identical graphs below.





2. Determine the domain of the following trigonometric identities:

a. $\cot(x) = \frac{\cos(x)}{\sin(x)}$ where the functions on both sides are defined.

This identity is defined only for $x \neq k\pi$, for all integers k.

b. $\cos(-u) = \cos(u)$ where the functions on both sides are defined.

This identity is defined for all real numbers u.

c. $\sec(y) = \frac{1}{\cos(y)}$ where the functions on both sides are defined.

This identity is defined for $y \neq \frac{\pi}{2} + k\pi$, for all integers k.

3. Rewrite $\sin(x)\cos^2(x) - \sin(x)$ as an expression containing a single term.

$$\begin{aligned}\sin(x) - \sin(x)\cos^2(x) &= \sin(x)\left(1 - \cos^2(x)\right) \\ &= \sin(x)\sin^2(x) \\ &= \sin^3(x)\end{aligned}$$

4. Suppose $0 < \theta < \frac{\pi}{2}$ and $\sin(\theta) = \frac{1}{\sqrt{3}}$. What is the value of $\cos(\theta)$?

$\cos(\theta) = \frac{\sqrt{6}}{3}$

5. If $\cos(\theta) = -\frac{1}{\sqrt{5}}$, what are possible values of $\sin(\theta)$?

Either $\sin(\theta) = \frac{2}{\sqrt{5}}$, or $\sin(\theta) = -\frac{2}{\sqrt{5}}$

6. Use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, where θ is any real number, to find the following:

a. $\cos(\theta)$, given $\sin(\theta) = \frac{5}{13}$, for $\frac{\pi}{2} < \theta < \pi$.

From the Pythagorean identity, $\cos^2(\theta) = 1 - \left(\frac{5}{13}\right)^2$. So, $\cos^2(\theta) = 1 - \frac{25}{169} = \frac{144}{169}$, and $\cos(\theta) = \frac{12}{13}$, or $\cos(\theta) = -\frac{12}{13}$. In the second quadrant, $\cos(\theta)$ is negative, so $\cos(\theta) = -\frac{12}{13}$.

b. $\tan(x)$, given $\cos(x) = -\frac{1}{\sqrt{2}}$, for $\pi < x < \frac{3\pi}{2}$.

From the Pythagorean identity, $\sin^2(x) = 1 - \left(\frac{1}{\sqrt{2}}\right)^2$. So, $\sin^2(x) = \frac{1}{2}$, and $\sin(x) = \frac{1}{\sqrt{2}}$, or $\sin(x) = -\frac{1}{\sqrt{2}}$.

Because $\sin(x)$ is negative in the third quadrant, $\tan(x) = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$.

7. The three identities below are all called Pythagorean identities. The second and third follow from the first, as you saw in Example 1 and the Exit Ticket.

   a. For which values of θ are each of these identities defined?

      i. $\sin^2(\theta) + \cos^2(\theta) = 1$, where the functions on both sides are defined.

         Defined for any real number θ.

      ii. $\tan^2(\theta) + 1 = \sec^2(\theta)$, where the functions on both sides are defined.

         Defined for real numbers θ such that $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

      iii. $1 + \cot^2(\theta) = \csc^2(\theta)$, where the functions on both sides are defined.

         Defined for real numbers θ such that $\theta \neq k\pi$, for all integers k.

   b. For which of the three identities is 0 in the domain of validity?

      Identities i and ii

   c. For which of the three identities is $\frac{\pi}{2}$ in the domain of validity?

      Identities i and iii

   d. For which of the three identities is $-\frac{\pi}{4}$ in the domain of validity?

      Identities i, ii, and iii

Lesson 16: Proving Trigonometric Identities

Student Outcomes

- Students prove simple identities involving the sine function, cosine function, and secant function.
- Students recognize features of proofs of identities.

Lesson Notes

Students find that in some circumstances, they can start with a false statement and logically arrive at a true statement; so, students should avoid beginning a proof with the statement to be proven. Instead, they should work on transforming one side of the equation into the other using only results that are known to be true. In this lesson, students prove several simple identities.

Classwork

Opening Exercise (10 minutes)

Have students work in pairs on the Opening Exercise.

MP.3

Opening Exercise

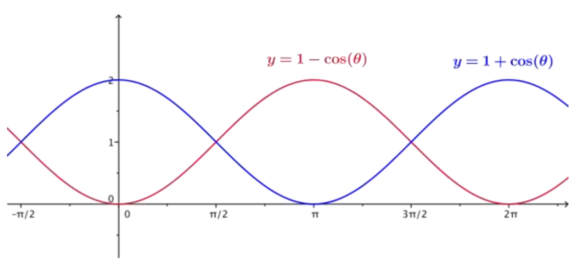
Which of these statements is a trigonometric identity? Provide evidence to support your claim.

Statement 1: $\sin^2(\theta) = 1 - \cos^2(\theta)$ **for** θ **any real number.**

Statement 2: $1 - \cos(\theta) = 1 - \cos(\theta)$ **for** θ **any real number.**

Statement 3: $1 - \cos(\theta) = 1 + \cos(\theta)$ **for** θ **any real number.**

Statement 1 is found by subtracting $\cos^2(\theta)$ from each side of the Pythagorean identity. Since the Pythagorean identity is true for all real numbers, Statement 1 is also an identity. The functions on either side of the equal sign in Statement 2 are identical, so they are also equivalent. The graphs of the functions on either side of the equal sign in Statement 3 intersect, as shown in the figure below, but the functions are not equivalent. Thus, Statements 1 and 2 are valid identities, but Statement 3 is false.



Using Statements 1 and 2, create a third identity, Statement 4, whose left side is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

Students are likely to divide the equation in Statement 1 by the equation in Statement 2 to get something like the following statement:

Statement 4: $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ *for all* θ *for which the functions on both sides are defined.*

For which values of θ is this statement valid?

The left side of the equation in Statement 4 is not defined when $\cos(\theta) = 1$, which happens when θ is a multiple of 2π. The right side of the equation is defined for all θ. Thus, the equation is true wherever both sides are defined, which is for all θ that are not multiples of 2π.

MP.3

Discuss in pairs what it might mean to "prove" an identity. What might it take to prove, for example, that the following statement is an identity?

$\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ **where $\theta \neq 2\pi k$, for all integers k.**

Students might come up with various suggestions, such as $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$, for $\theta \neq 2\pi k$ for all integers k is an identity if $\frac{\sin^2(\theta)}{1-\cos(\theta)}$ and $1 + \cos(\theta)$ define the same functions or if they take the same values for the variable θ. The functions on the two sides should have the same domain, but that alone is not enough to make the statement an identity.

To prove an identity, you have to use logical steps to show that one side of the equation in the identity can be transformed into the other side of the equation using already established identities such as the Pythagorean identity or the properties of operation (commutative, associative, and distributive properties). It is not correct to start with what you want to prove and work on both sides of the equation at the same time, as the following exercise shows.

Scaffolding:

- Ask students struggling to see that these are identities to substitute several values for θ into the left and the right sides of these equations separately to verify that they are true equations.
- Demonstrate how to determine Statement 4 for students having trouble seeing it.
- To challenge students, ask them to generate another identity using Statements 1 and 2 and explain for which values of θ it is valid.

Scaffolding:

Student pairs may need to first discuss what it means to prove anything. Circulate to assist those having trouble with the question and to find those who might present their answer.

Exercise 1 (8 minutes)

Students should work on this exercise in groups. Its purpose is to show that if they start with the goal of a proof, they can end up "proving" a statement that is false. Part of standard MP.3 involves distinguishing correct reasoning from flawed reasoning, and if there is a flaw in the argument, explaining what the flaw is. In this exercise, students see how a line of reasoning can go wrong.

Begin by asking students to take out their calculators and quickly graph the functions $f(x) = \sin(x) + \cos(x)$ and $g(x) = -\sqrt{1 + 2\sin(x)\cos(x)}$ to determine whether $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for all θ for which both functions are defined is a valid identity. Students should see from the graphs that the functions are not equivalent.

Scaffolding:

If students have trouble seeing the problem here, ask them to consider the following similar argument:

- [a] $1 = (-1)$, so squaring each side, we get

  [b] $1 = 1$, which is an identity.
- Therefore, squaring each side of a false statement can yield an identity. That does not make the original statement true.

Exercise 1

1. Use a calculator to graph the functions $f(x) = \sin(x) + \cos(x)$ and $g(x) = -\sqrt{1 + 2\sin(x)\cos(x)}$ to determine whether $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for all θ for which both functions are defined is a valid identity. You should see from the graphs that the functions are not equivalent.

   Suppose that Charles did not think to graph the equations to see if the given statement was a valid identity, so he set about proving the identity using algebra and a previous identity. His argument is shown below.

   First, [1] $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for θ any real number.

   Now, using the multiplication property of equality, square both sides, which gives

   [2] $\sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = 1 + 2\sin(\theta)\cos(\theta)$ for θ any real number.

   Using the subtraction property of equality, subtract $2\sin(\theta)\cos(\theta)$ from each side, which gives

   [3] $\sin^2(\theta) + \cos^2(\theta) = 1$ for θ any real number.

   Statement [3] is the Pythagorean identity. So, replace $\sin^2(\theta) + \cos^2(\theta)$ by 1 to get

   [4] $1 = 1$, which is definitely true.

   Therefore, the original statement must be true.

   Does this mean that Charles has proven that statement [1] is an identity? Discuss with your group whether it is a valid proof. If you decide it is not a valid proof, then discuss with your group how and where his argument went wrong.

   No, statement [1] is not an identity; in fact, it is not true, as we showed above by graphing the functions on the two sides of the equation. The sequence of statements is not a proof because it starts with a false statement in statement [1]. Squaring both sides of the equation is an irreversible step that alters the solutions to the equation. When squaring both sides of an equation, we have assumed that the equality exists, and that amounts to assuming what one is trying to prove. A better approach to prove an identity is valid would be to take one side of the equation in the proposed identity and work on it until one gets the other side.

The logic used by Charles is essentially, "If Statement [1] is true, then Statement [1] is true," which does not establish that Statement [1] is true. Make sure that students understand that all statements in a proof, particularly the first step of a proof, must be known to be true and must follow logically from the preceding statements in order for the proof to be valid.

Example 1 (10 minutes): Two Proofs of Our New Identity

Example 1: Two Proofs of Our New Identity

Work through these two different ways to approach proving the identity $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ where $\theta \neq 2\pi k$, for integers k. The proofs make use of some of the following properties of equality and real numbers. Here a, b, and c stand for arbitrary real numbers.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \cdot c = b \cdot c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.
Associative properties	$(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
Commutative properties	$a + b = b + a$ and $ab = ba$.
Distributive property	$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Fill in the missing parts of the proofs outlined in the tables below. Then, write a proof of the resulting identity.

a. **We start with the Pythagorean identity. When we divide both sides by the same expression, $1 - \cos(\theta)$, we introduce potential division by zero when $\cos(\theta) = 1$. This will change the set of values of θ for which the identity is valid.**

PROOF:

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\sin^2(\theta) + \cos^2(\theta)$	$=$	1	θ any real number	Pythagorean identity
2	$\sin^2(\theta)$	$=$	$1 - \cos^2(\theta)$	θ any real number	*Subtraction property of equality*
3	$\sin^2(\theta)$	$=$	$(1 - \cos(\theta))(1 + \cos(\theta))$	θ any real number	*Distributive property*
4	$\frac{\sin^2(\theta)}{1 - \cos(\theta)}$	$=$	$\frac{(1 - \cos(\theta))(1 + \cos(\theta))}{1 - \cos(\theta)}$	*$\theta \neq 2\pi k$ for all integers k*	*Division property of equality*
5	$\frac{\sin^2(\theta)}{1 - \cos(\theta)}$	$=$	$1 + \cos(\theta)$	$\theta \neq 2\pi k$ for all integers k	Substitution property of equality using $\frac{1 - \cos(\theta)}{1 - \cos(\theta)} = 1$

Let θ be a real number so that $\theta \neq 2\pi k$, for all integers k. Then, $\cos(\theta) \neq 1$, so $1 - \cos(\theta) \neq 0$. By the Pythagorean Identity, $\sin^2(\theta) + \cos^2(\theta) = 1$. Then, $\sin^2(\theta) = 1 - \cos^2(\theta)$, and we can divide both sides by $1 - \cos(\theta)$ to give $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{1-\cos^2(\theta)}{1-\cos(\theta)}$. Factoring the numerator of the right side, we have $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$; thus, $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ where $\theta \neq 2\pi k$, for all integers k.

b. Or, we can start with the more complicated side of the identity we want to prove and use algebra and prior trigonometric definitions and identities to transform it to the other side. In this case, the more complicated expression is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

PROOF:

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$\frac{1-\cos^2(\theta)}{1-\cos(\theta)}$	$\theta \neq 2\pi k$, for all integers k	Substitution property of equality using $\sin^2(\theta) = 1-\cos^2(\theta)$
2	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$\frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$	*$\theta \neq 2\pi k$, for all integers k*	Distributive property
3	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$1+\cos(\theta)$	*$\theta \neq 2\pi k$, for all integers k*	*Substitution property of equality using $\frac{1-\cos(\theta)}{1-\cos(\theta)} = 1$*

Let θ be a real number so that $\theta \neq 2\pi k$, for all integers k. We begin with the expression $\frac{\sin^2(\theta)}{1-\cos(\theta)}$. According to the Pythagorean identity, $\sin^2(\theta) = 1-\cos^2(\theta)$. Substituting this into our expression, we have $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{1-\cos^2(\theta)}{1-\cos(\theta)}$. Factoring the numerator gives $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$. Thus, $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1+\cos(\theta)$ where $\theta \neq 2\pi k$, for all integers k.

Exercises 2–3 (12 minutes)

Students should work on these exercises individually and then share their results either in a group or with the whole class. Before beginning to prove an identity, students might want to take some scratch paper and work out the main ideas of the proof, taking into account the values for which the functions on either side of the equation are not defined. Then, they can restrict the values of x or θ at the beginning of the proof and not have to worry about it at every step.

Scaffolding:

While students work independently on these exercises, work with a small group that would benefit from more teacher modeling in a small group setting.

Exercises 2–3

Prove that the following are trigonometric identities, beginning with the side of the equation that seems to be more complicated and starting the proof by restricting x to values where the identity is valid. Make sure that the complete identity statement is included at the end of the proof.

2. $\tan(x) = \frac{\sec(x)}{\csc(x)}$ **for real numbers** $x \neq \frac{\pi}{2} + \pi k$**, for all integers** k**.**

The more complicated side of the equation is $\frac{\sec(x)}{\csc(x)}$*, so we begin with it. First, we eliminate values of* x *that are not in the domain.*

PROOF: Let x *be a real number so that* $x \neq \frac{\pi}{2} + \pi k$*, for all integers* k*.*

Applying the definitions of the secant and cosecant functions, we have

$$\frac{\sec(x)}{\csc(x)} = \frac{\frac{1}{\cos(x)}}{\frac{1}{\sin(x)}}.$$

Simplifying the complex fraction gives

$$\begin{aligned}\frac{\sec(x)}{\csc(x)} &= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{1} \\ &= \frac{\sin(x)}{\cos(x)} \\ &= \tan(x).\end{aligned}$$

Thus, $\tan(x) = \frac{\sec(x)}{\csc(x)}$ *for* $x \neq \frac{\pi}{2} + \pi k$*, for all integers* k*.*

3. $\cot(x) + \tan(x) = \sec(x)\csc(x)$ **for all real numbers** $x \neq \frac{\pi}{2}n$ **for integer** n**.**

The sides seem equally complicated, but the left side has two terms, so we begin with it. In general, functions composed of multiple terms (or a product of multiple terms) can be seen as more complicated than functions having a single term. However, a valid proof can be written starting on either side of the equation.

PROOF: Let x *be a real number so that* $x \neq \frac{\pi}{2}k$*, for all integers* k*. Then, we express* $\cot(x) + \tan(x)$ *in terms of* $\sin(x)$ *and* $\cos(x)$ *and find a common denominator.*

$$\begin{aligned}\cot(x) + \tan(x) &= \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos^2(x)}{\sin(x)\cos(x)} + \frac{\sin^2(x)}{\sin(x)\cos(x)}.\end{aligned}$$

Adding and applying the Pythagorean identity and then converting to the secant and cotangent functions gives

$$\begin{aligned}\cot(x) + \tan(x) &= \frac{1}{\sin(x)\cos(x)} \\ &= \frac{1}{\sin(x)} \cdot \frac{1}{\cos(x)} \\ &= \csc(x)\sec(x).\end{aligned}$$

Therefore, $\cot(x) + \tan(x) = \sec(x)\csc(x)$*, where* $x \neq \frac{\pi}{2}n$ *for all integers* n*.*

Closing (1 minute)

Ask students to explain how to prove a trigonometric identity, either in writing, to a partner, or as a class. Some key points they should mention are listed below.

- Start with stating the values of the variable—usually x or θ—for which the identity is valid.
- Work from a fact that is known to be true from either a prior identity or algebraic fact.
- A general plan is to start with the more complicated side of the equation you are trying to establish and transform it using a series of steps that can each be justified by prior facts and rules of algebra. The goal is to create a sequence of equations that are logically equivalent and that end with the desired equation for your identity.
- Note that we cannot start with the equation we want to establish because that is assuming what we are trying to prove. "If A is true, then A is true" does not logically establish that A is true.

Exit Ticket (4 minutes)

Name ______________________________ Date ________________

Lesson 16: Proving Trigonometric Identities

Exit Ticket

Prove the following identity:

$\tan(\theta)\sin(\theta) + \cos(\theta) = \sec(\theta)$ for real numbers θ, where $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k.

Exit Ticket Sample Solutions

Prove the following identity:

$$\tan(\theta)\sin(\theta) + \cos(\theta) = \sec(\theta) \text{ for real numbers } \theta, \text{ where } \theta \neq \frac{\pi}{2} + \pi k, \text{ for all integers } k.$$

Begin with the more complicated side. Find a common denominator, use the Pythagorean identity, and then convert the fraction to its reciprocal.

PROOF: Let θ be any real number so that $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k. Then,

$$\begin{aligned}\tan(\theta)\sin(\theta) + \cos(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}\sin(\theta) + \cos(\theta) \\ &= \frac{\sin^2(\theta)}{\cos(\theta)} + \frac{\cos^2(\theta)}{\cos(\theta)} \\ &= \frac{1}{\cos(\theta)} \\ &= \sec(\theta).\end{aligned}$$

Therefore, $\tan(\theta)\sin(\theta) + \cos(\theta) = \sec(\theta)$, where $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k.

Problem Set Sample Solutions

The first problem is designed to reinforce that the sine function is not linear. The addition formulas for $\sin(x+y)$ and $\cos(x+y)$ are introduced in the next lesson.

1. **Does $\sin(x+y)$ equal $\sin(x) + \sin(y)$ for all real numbers x and y?**

   a. **Find each of the following: $\sin\left(\frac{\pi}{2}\right)$, $\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{4}\right)$.**

   $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *and* $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

   b. **Are $\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{4}\right)$ equal?**

   No, because $\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *and* $\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{4}\right) = 1 + \frac{\sqrt{2}}{2}$.

   c. **Are there any values of x and y for which $\sin(x+y) = \sin(x) + \sin(y)$?**

   Yes. If either x or y is zero, or if both x and y are multiples of π, this is a true statement. In many other cases it is not true, so it is not true in general.

2. Use $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and identities involving the sine and cosine functions to establish the following identities for the tangent function. Identify the values of x where the equation is an identity.

a. $\tan(\pi - x) = \tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)} = \frac{\sin(x)}{-\cos(x)}$$
$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Thus, $\tan(\pi - x) = -\tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

b. $\tan(x + \pi) = \tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin(x)}{-\cos(x)}$$
$$= \frac{\sin(x)}{\cos(x)} = \tan(x).$$

Thus, $\tan(x + \pi) = \tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

c. $\tan(2\pi - x) = -\tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(2\pi - x) = \frac{\sin(2\pi - x)}{\cos(2\pi - x)} = \frac{-\sin(x)}{\cos(x)}$$
$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Thus, $\tan(2\pi - x) = -\tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

d. $\tan(-x) = -\tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)}$$
$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Thus, $\tan(-x) = -\tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

3. Rewrite each of the following expressions as a single term. Identify the values of x for which the original expression and your expression are equal:

a. $\cot(x)\sec(x)\sin(x)$

$$\begin{aligned}\cot(x)\sec(x)\sin(x) &= \frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\cos(x)}\cdot\frac{\sin(x)}{1}\\ &= 1.\end{aligned}$$

The expressions are equal where $x \neq \frac{\pi}{2}k$, for all integers k.

b. $\left(\frac{1}{1-\sin(x)}\right)\left(\frac{1}{1+\sin(x)}\right)$

$$\begin{aligned}\left(\frac{1}{1-\sin(x)}\right)\left(\frac{1}{1+\sin(x)}\right) &= \frac{1}{1-\sin^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x).\end{aligned}$$

The expressions are equal where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

c. $\frac{1}{\cos^2(x)} - \frac{1}{\cot^2(x)}$

$$\begin{aligned}\frac{1}{\cos^2(x)} - \frac{1}{\cot^2(x)} &= \frac{1}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)}\\ &= \frac{1-\sin^2(x)}{\cos^2(x)}\\ &= 1.\end{aligned}$$

The expressions are equal where $x \neq \frac{\pi}{2}k$, for all integers k.

d. $\frac{(\tan(x)-\sin(x))(1+\cos(x))}{\sin^3(x)}$

$$\begin{aligned}\frac{(\tan(x)-\sin(x))(1+\cos(x))}{\sin^3(x)} &= \frac{\left(\frac{\sin(x)}{\cos(x)} - \sin(x)\right)(1+\cos(x))}{\sin^3(x)}\\ &= \frac{\sin(x)\left(\frac{1-\cos(x)}{\cos(x)}\right)(1+\cos(x))}{\sin^3(x)}\\ &= \frac{\left(\frac{1-\cos^2(x)}{\cos(x)}\right)}{\sin^2(x)}\\ &= \frac{1}{\cos(x)}\\ &= \sec(x).\end{aligned}$$

The expressions are equal where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

4. Prove that for any two real numbers a and b,

$$\sin^2(a) - \sin^2(b) + \cos^2(a)\sin^2(b) - \sin^2(a)\cos^2(b) = 0.$$

PROOF: Let a and b be any real numbers. Then,

$$\begin{aligned}\sin^2(a) - \sin^2(b) + \cos^2(a)\sin^2(b) - \sin^2(a)\cos^2(b) &= \sin^2(a)\big(1-\cos^2(b)\big) - \sin^2(b)\big(1-\cos^2(a)\big)\\ &= \sin^2(a)\sin^2(b) - \sin^2(b)\sin^2(a)\\ &= 0\end{aligned}$$

Therefore, $\sin^2(a) - \sin^2(b) + \cos^2(a)\sin^2(b) - \sin^2(a)\cos^2(b) = 0$, for all real numbers a and b.

5. Prove that the following statements are identities for all values of θ for which both sides are defined, and describe that set.

a. $\cot(\theta)\sec(\theta) = \csc(\theta)$

PROOF: Let θ be a real number so that $\theta \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\cot(\theta)\sec(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{1}{\cos(\theta)}\\ &= \frac{1}{\sin(\theta)}\\ &= \csc(\theta).\end{aligned}$$

Therefore, $\cot(\theta)\sec(\theta) = \csc(\theta)$, for all values of $\theta \neq \frac{\pi}{2}k$ for all integers k.

b. $\big(\csc(\theta) + \cot(\theta)\big)\big(1 - \cos(\theta)\big) = \sin(\theta)$

PROOF: Let θ be a real number so that $\theta \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\big(\csc(\theta) + \cot(\theta)\big)\big(1 - \cos(\theta)\big) &= \frac{1+\cos(\theta)}{\sin(\theta)}\big(1-\cos(\theta)\big)\\ &= \frac{1-\cos^2(\theta)}{\sin(\theta)}\\ &= \frac{\sin^2(\theta)}{\sin(\theta)}\\ &= \sin(\theta).\end{aligned}$$

Therefore, $\big(\csc(\theta) + \cot(\theta)\big)\big(1 - \cos(\theta)\big) = \sin(\theta)$, for all values of $\theta \neq \frac{\pi}{2}k$ for any integer k.

c. $\tan^2(\theta) - \sin^2(\theta) = \tan^2(\theta)\sin^2(\theta)$

PROOF: Let θ be a real number so that $\theta \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\tan^2(\theta) - \sin^2(\theta) &= \frac{\sin^2(\theta)}{\cos^2(\theta)} - \sin^2(\theta)\\ &= \frac{\sin^2(\theta)(1-\cos^2(\theta))}{\cos^2(\theta)}\\ &= \frac{\sin^2(\theta)\sin^2(\theta)}{\cos^2(\theta)}\\ &= \tan^2(\theta)\sin^2(\theta).\end{aligned}$$

Therefore, then $\tan^2(\theta) - \sin^2(\theta) = \tan^2(\theta)\sin^2(\theta)$, for all values of $\theta \neq \frac{\pi}{2}k$ for any integer k.

d. $\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)} = 3\sin^2(x)$

PROOF: Let x be a real number so that $x \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)} &= \frac{4+\frac{\sin^2(x)}{\cos^2(x)}-\frac{1}{\cos^2(x)}}{\frac{1}{\sin^2(x)}}\\ &= \frac{\frac{4\cos^2(x)+\sin^2(x)-1}{\cos^2(x)}}{\frac{1}{\sin^2(x)}}\\ &= \frac{\sin^2(x)(4\cos^2(x)-\cos^2(x))}{\cos^2(x)}\\ &= 3\sin^2(x).\end{aligned}$$

Therefore $\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)} = 3\sin^2(x)$, for all $x \neq \frac{\pi}{2}k$ for any integer k.

e. $\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)} = 2$

PROOF: Let θ be a real number so that $\theta \neq -\frac{\pi}{2}+2k\pi$, for all integers k. Then,

$$\begin{aligned}\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)} &= \frac{1+2\sin(\theta)+\sin^2(\theta)+\cos^2(\theta)}{1+\sin(\theta)}\\ &= \frac{2+2\sin(\theta)}{1+\sin(\theta)}\\ &= 2.\end{aligned}$$

Therefore, $\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)} = 2$, for all $\theta \neq -\frac{\pi}{2}+2k\pi$, for any integer k.

6. Prove that the value of the following expression does not depend on the value of y:

$$\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}.$$

If $y \neq \frac{\pi}{2}+k\pi$ for all integers k, then

$$\begin{aligned}\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)} &= \frac{\cos(y)}{\sin(y)}\cdot\frac{\frac{\sin(x)}{\cos(x)}+\frac{\sin(y)}{\cos(y)}}{\frac{\cos(x)}{\sin(x)}+\frac{\cos(y)}{\sin(y)}}\\ &= \frac{\cos(y)}{\sin(y)}\cdot\frac{\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)}}{\frac{\cos(x)\sin(y)+\sin(x)\cos(y)}{\sin(x)\sin(y)}}\\ &= \frac{\cos(y)}{\sin(y)}\cdot\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}\\ &= \frac{\sin(x)}{\cos(x)}\\ &= \tan(x).\end{aligned}$$

Therefore, $\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)} = \tan(x)$ for all values of x and y for which both sides of the equation are defined. Thus, the expression does not depend on the value of y.

Lesson 17: Trigonometric Identity Proofs

Student Outcomes

- Students see derivations and proofs of the addition and subtraction formulas for sine and cosine.
- Students prove some simple trigonometric identities.

Lesson Notes

The lesson starts with students looking for patterns in a table to make conjectures about the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$. From these formulas, students can quickly deduce the formulas for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$. The teacher gives proofs of important formulas, and then students prove some simple trigonometric identities. The lesson highlights MP.3 and MP.8, as students look for patterns in repeated calculations and construct arguments about the patterns they find.

Scaffolding:

- Students should have access to calculators.
- The teacher may want to model a few calculations at the beginning.
- Pairs of students who fill in the table quickly should be encouraged to help those who might be struggling.
- Once a few students have filled in the table, one or two might be encouraged to share their entries with the class because when all students have the entries, they are in a better position to discover the rule.

Classwork

Opening Exercise (10 minutes)

MP.8 Students should work in pairs to fill out the table and look for patterns. They should be looking for columns whose entries might be combined to yield the entries in the column for $\sin(\alpha + \beta)$.

Opening Exercise

We have seen that $\sin(\alpha+\beta) \neq \sin(\alpha) + \sin(\beta)$. So, what is $\sin(\alpha+\beta)$? Begin by completing the following table:

α	β	$\sin(\alpha)$	$\sin(\beta)$	$\sin(\alpha+\beta)$	$\sin(\alpha)\cos(\beta)$	$\sin(\alpha)\sin(\beta)$	$\cos(\alpha)\cos(\beta)$	$\cos(\alpha)\sin(\beta)$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$
$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}+\sqrt{6}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{2}}{4}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}+\sqrt{6}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$

MP.8 Ask students to write an equation that describes how the entries in other columns might be combined to yield the entries in the shaded column.

The identity they are looking for in the table is the following:

$$\sin(\alpha+\beta)=\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta).$$

Emphasize in the discussion that the proposed identity has not been proven; it has only been tested for some specific values of α and β. Its status now is as a conjecture.

The conjecture is strengthened by the following observation: Because α and β play the same role in $(\alpha+\beta)$, they should not play different roles in any formula for the sine of that sum. In the conjecture, if α and β are interchanged, the formula remains essentially the same. That symmetry helps make the conjecture more plausible.

Scaffolding:

- If no student offers the identity, the teacher might suggest that students look for two columns whose entries sum to yield $\sin(\alpha+\beta)$.
- It might even be necessary for the teacher to point at the two columns.

Use the following table to formulate a conjecture for $\cos(\alpha+\beta)$:

α	β	$\cos(\alpha)$	$\cos(\beta)$	$\cos(\alpha+\beta)$	$\sin(\alpha)\cos(\beta)$	$\sin(\alpha)\sin(\beta)$	$\cos(\alpha)\cos(\beta)$	$\cos(\alpha)\sin(\beta)$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$
$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{2}}{4}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}-\sqrt{6}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{6}}{4}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{4}$

MP.8 Again, ask students to write an equation that describes how the entries in other columns might be combined to yield the entries in the shaded column.

The identity they are looking for in the table is the following:

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

Again, in a class discussion of the exercise after students have looked for a pattern, if no student comes up with that identity, the teacher may want to point at the two columns whose entries differ to yield $\cos(\alpha+\beta)$. It should be repeated that the proposed identity has not been proven; it has only been tested for some specific values of α and β. It is a conjecture.

This conjecture, too, is strengthened by the observation that because α and β play the same role in $(\alpha+\beta)$, they should not play different roles in any formula for the cosine of that sum. And again, if α and β are interchanged in the conjectured formula, it remains essentially the same. That symmetry helps make the conjecture more plausible.

Examples 1–2 (15 minutes): Formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$

Scaffolding:

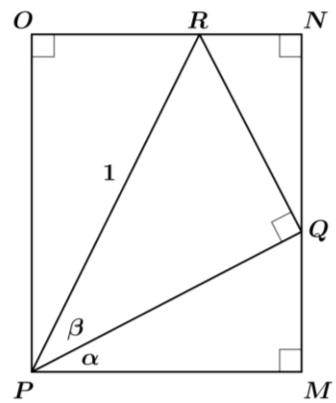
Because the proofs of the addition and subtraction formulas for sine and cosine can be complicated, only the proof of the sine addition formula is presented in detail here. Advanced students might be asked to prove, in much the same fashion, any of the other three formulas, as well as by deriving them from the sine addition formula. Problem 1 in the Problem Set concerns one of those proofs.

Examples 1–2: Formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$

1. **One conjecture is that the formula for the sine of the sum of two numbers is $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. The proof can be a little long, but it is fairly straightforward. We will prove only the case when the two numbers are positive, and their sum is less than $\frac{\pi}{2}$.**

   a. **Let α and β be positive real numbers such that $0 < \alpha + \beta < \frac{\pi}{2}$.**

   b. **Construct rectangle $MNOP$ such that $PR = 1$, $m\angle PQR = 90°$, $m\angle RPQ = \beta$, and $m\angle QPM = \alpha$. See the figure on the right.**



   c. **Fill in the blanks in terms of α and β:**

      i. $m\angle RPO =$ ______________.

      $\frac{\pi}{2} - \alpha - \beta$

      ii. $m\angle PRO =$ ______________.

      $\alpha + \beta$

      iii. Therefore, $\sin(\alpha + \beta) = PO$.

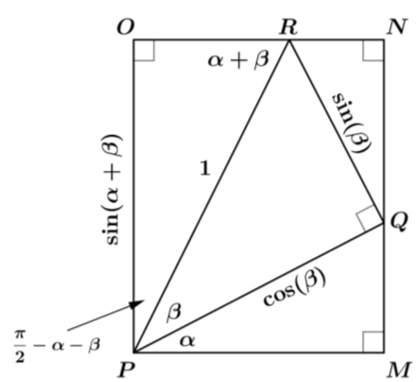
      iv. $RQ = \sin($_____$)$.

      β

      v. $PQ = \cos($_____$)$.

      β

   d. **Let's label the angle and length measurements as shown.**



   e. **Use this new figure to fill in the blanks in terms of α and β:**

      i. Why does $\sin(\alpha) = \frac{MQ}{\cos(\beta)}$?

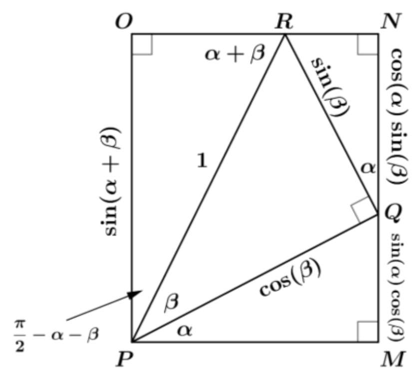
      The length of the hypotenuse of $\triangle PQM$ is $\cos(\beta)$, and MQ is the length of the side opposite α.

      ii. Therefore, $MQ =$ ___________.

      $\sin(\alpha)\cos(\beta)$

      iii. $m\angle RQN =$ ____________.

      α



f. Now, consider $\triangle RQN$. Since $\cos(\alpha) = \frac{QN}{\sin(\beta)}$,

i. $QN =$ ______________.

$\cos(\alpha)\sin(\beta)$

2.

a. Label these lengths and angle measurements in the figure.

b. Since $MNOP$ is a rectangle, $OP = MQ + QN$.

c. Thus, $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

Note that we have only proven the formula for the sine of the sum of two real numbers α and β in the case where $0 < \alpha + \beta < \frac{\pi}{2}$. A proof for all real numbers α and β breaks down into cases that are proven similarly to the case we have just seen. Although we are omitting the full proof, this formula holds for all real numbers α and β.

For any real numbers α and β,

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

Scaffolding:

A wall poster with all four sum and difference formulas will help students keep these formulas straight.

3. Now, let's prove our other conjecture, which is that the formula for the cosine of the sum of two numbers is

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Again, we will prove only the case when the two numbers are positive, and their sum is less than $\frac{\pi}{2}$. This time, we will use the sine addition formula and identities from previous lessons instead of working through a geometric proof.

Fill in the blanks in terms of α and β:

Let α and β be any real numbers. Then,

$$\begin{aligned} \cos(\alpha + \beta) &= \sin\left(\frac{\pi}{2} - (\_\_\_\_\_\_\_)\right) \\ &= \sin((\_\_\_\_\_\_\_) - \beta) \\ &= \sin((\_\_\_\_\_\_\_) + (-\beta)) \\ &= \sin(\_\_\_\_\_\_\_)\cos(-\beta) + \cos(\_\_\_\_\_\_\_)\sin(-\beta) \\ &= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta). \end{aligned}$$

The completed proof should look like the following:

$$\begin{aligned} \cos(\alpha + \beta) &= \sin\left(\frac{\pi}{2} - (\alpha + \beta)\right) \\ &= \sin\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right) \\ &= \sin\left(\left(\frac{\pi}{2} - \alpha\right) + (-\beta)\right) \\ &= \sin\left(\frac{\pi}{2} - \alpha\right)\cos(-\beta) + \cos\left(\frac{\pi}{2} - \alpha\right)\sin(-\beta) \\ &= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta). \end{aligned}$$

For all real numbers α and β,

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Exercises 1–2 (6 minutes): Formulas for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$

In these exercises, formulas for the sine and cosine of the difference of two angles are developed from the formulas for the sine and cosine of the sum of two angles.

Exercises 1–2: Formulas for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$

1. **Rewrite the expression $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$. Use the rewritten form to find a formula for the sine of the difference of two angles, recalling that the sine is an odd function.**

   Let α and β be any real numbers. Then,

   $$\begin{aligned}\sin(\alpha+(-\beta)) &= \sin(\alpha)\cos(-\beta)+\cos(\alpha)\sin(-\beta)\\ &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta).\end{aligned}$$

   Therefore, $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$ for all real numbers α and β.

2. **Now, use the same idea to find a formula for the cosine of the difference of two angles. Recall that the cosine is an even function.**

   Let α and β be any real numbers. Then,

   $$\begin{aligned}\cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos(\alpha)\cos(-\beta)-\sin(\alpha)\sin(-\beta)\\ &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta).\end{aligned}$$

   Therefore, $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ for all real numbers α and β.

For all real numbers α and β,

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta), \text{ and}$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

Scaffolding:

- To help students understand the difference formulas, consider giving them some examples to calculate.
- $$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{2}\sqrt{3}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{2}(\sqrt{3}-1)}{4}\end{aligned}$$
- $$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{2}\sqrt{3}}{4}+\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{2}(\sqrt{3}+1)}{4}\end{aligned}$$

Exercises 3–5 (10 minutes)

These exercises make use of the formulas proved in the examples. Students should work on these exercises in pairs.

Use the sum and difference formulas to do the following:

Exercises 3–5

3. **Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$, where all of the expressions are defined. Hint: Use the addition formulas for sine and cosine.**

   Let α and β be any real numbers so that $\cos(\alpha) \neq 0$, $\cos(\beta) \neq 0$, and $\cos(\alpha+\beta) \neq 0$. By the definition of tangent, $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$.

   Using sum formulas for sine and cosine, we have

   $$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}.$$

   Dividing numerator and denominator by $\cos(\alpha)\cos(\beta)$ gives

   $$\frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

   Therefore, $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$ for any real numbers α and β so that $\cos(\alpha) \neq 0$, $\cos(\beta) \neq 0$, and $\cos(\alpha+\beta) \neq 0$.

Scaffolding:
Students may need to be prompted to divide the numerator and denominator by $\cos(\alpha)\cos(\beta)$.

4. **Derive a formula for $\sin(2u)$ in terms of $\sin(u)$ and $\cos(u)$ for all real numbers u.**

   Let u be any real number. Then, $\sin(2u) = \sin(u+u) = \sin(u)\cos(u) + \cos(u)\sin(u)$, which is equivalent to $\sin(2u) = 2\sin(u)\cos(u)$.

   Therefore, $\sin(2u) = 2\sin(u)\cos(u)$ for all real numbers u.

5. **Derive a formula for $\cos(2u)$ in terms of $\sin(u)$ and $\cos(u)$ for all real numbers u.**

   Let u be a real number. Then, $\cos(2u) = \cos(u+u) = \cos(u)\cos(u) - \sin(u)\sin(u)$, which is equivalent to $\cos(2u) = \cos^2(u) - \sin^2(u)$.

   Therefore, $\cos(2u) = \cos^2(u) - \sin^2(u)$ for all real numbers u. Using the Pythagorean identities, you can rewrite this identity as $\cos(2u) = 2\cos^2(u) - 1$ or as $\cos(2u) = 1 - 2\sin^2(u)$ for all real numbers u.

Closing (1 minute)

Ask students to respond to this question in writing, to a partner, or as a class.

- Edna claims that in the same way that $2(\alpha + \beta) = 2(\alpha) + 2(\beta)$, it follows by the distributive property that $\sin(\alpha + \beta) = \sin(\alpha) + \sin(\beta)$ for all real numbers α and β. Danielle says that can't be true. Who is correct, and why?
    - *Danielle is correct. Given that* $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, *it follows that* $\sin(\alpha + \beta) = \sin(\alpha) + \sin(\beta)$ *only for special values of* α *and* β. *That is, when* $\cos(\beta) = 1$ *and* $\cos(\alpha) = 1$, *or when* $\alpha = \beta = \pi n$ *for* n *an integer. So, in general,*

      $$\sin(\alpha + \beta) \neq \sin(\alpha) + \sin(\beta).$$

      A simple example is when $\alpha = \beta = \frac{\pi}{2}$. *Then,* $\sin(\alpha + \beta) = \sin(\pi) = 0$, *but* $\sin(\alpha) + \sin(\beta) = \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 2$. Since $0 \neq 2$, $\sin(\alpha + \beta)$ is generally not equal to $\sin(\alpha) + \sin(\beta)$.

Exit Ticket (3 minutes)

Name ____________________ Date ____________

Lesson 17: Trigonometric Identity Proofs

Exit Ticket

Derive a formula for $\tan(\alpha - \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$, where $\alpha \neq \frac{\pi}{2} + k\pi$ and $\beta \neq \frac{\pi}{2} + k\pi$, for all integers k.

Exit Ticket Sample Solutions

Derive a formula for $\tan(\alpha-\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$, where $\alpha \neq \frac{\pi}{2}+k\pi$ and $\beta \neq \frac{\pi}{2}+k\pi$, for all integers k.

Let α and β be real numbers so that $\alpha \neq \frac{\pi}{2}+k\pi$ and $\beta \neq \frac{\pi}{2}+k\pi$, for all integers k. Using the definition of tangent, $\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$.

Using the difference formulas for sine and cosine,

$$\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)}.$$

Dividing numerator and denominator by $\cos(\alpha)\cos(\beta)$ gives

$$\frac{\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)}=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}.$$

Therefore, $\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$, where $\alpha \neq \frac{\pi}{2}+k\pi$ and $\beta \neq \frac{\pi}{2}+k\pi$, for all integers k.

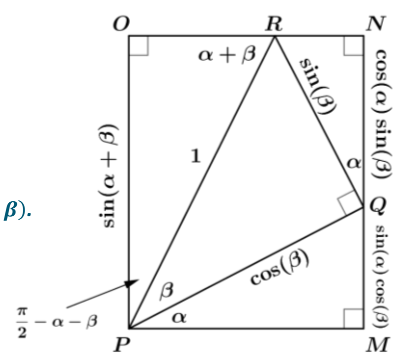
Problem Set Sample Solutions

These problems continue the derivation and demonstration of simple trigonometric identities.

1. Prove the formula

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \text{ for } 0<\alpha+\beta<\frac{\pi}{2}$$

using the rectangle $MNOP$ in the figure on the right and calculating PM, RN, and RO in terms of α and β.



PROOF: Let α and β be real numbers so that $0<\alpha+\beta<\frac{\pi}{2}$.

Then $PM=\cos(\alpha)\cos(\beta)$, $RN=\sin(\alpha)\sin(\beta)$, and $RO=\cos(\alpha+\beta)$.

Because $RO=PM-RN$, it follows that

$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$ for $0<\alpha+\beta<\frac{\pi}{2}$.

2. Derive a formula for $\tan(2u)$ for $u \neq \frac{\pi}{4}+\frac{k\pi}{2}$ and $u \neq \frac{\pi}{2}+k\pi$, for all integers k.

PROOF: Let u be any real number so that $u \neq \frac{\pi}{4}+\frac{k\pi}{2}$, and $u \neq \frac{\pi}{2}+k\pi$, for all integers k. In the formula $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$, replace α and β both by u. The resulting equation is $\tan(2u)=\frac{\tan(u)+\tan(u)}{1-\tan(u)\tan(u)}$, which is equivalent to

$$\tan(2u)=\frac{2\tan(u)}{1-\tan^2(u)} \text{ for } u \neq \frac{\pi}{4}+\frac{k\pi}{2} \text{ and } u \neq \frac{\pi}{2}+k\pi, \text{ for all integers } k.$$

3. Prove that $\cos(2u)=2\cos^2(u)-1$ for any real number u.

PROOF: Let u be any real number. From Exercise 3 in class, we know that $\cos(2u)=\cos^2(u)-\sin^2(u)$ for any real number u. Using the Pythagorean identity, we know that $\sin^2(u)=1-\cos^2(u)$. By substitution, $\cos(2u)=\cos^2(u)-1+\cos^2(u)$.

Thus, $\cos(2u)=2\cos^2(u)-1$ for any real number u.

4. Prove that $\frac{1}{\cos(x)} - \cos(x) = \sin(x) \cdot \tan(x)$ for $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

We begin with the left side, get a common denominator, and then use the Pythagorean identity.

PROOF: Let x be a real number so that $x \neq \frac{\pi}{2} + k\pi$, for all integers k. Then,

$$\begin{aligned}\frac{1}{\cos(x)} - \cos(x) &= \frac{1 - \cos^2(x)}{\cos(x)} \\ &= \frac{\sin^2(x)}{\cos(x)} \\ &= \frac{\sin(x)}{\cos(x)} \cdot \sin(x) \\ &= \sin(x) \cdot \tan(x).\end{aligned}$$

Therefore, $\frac{1}{\cos(x)} - \cos(x) = \sin(x) \cdot \tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

5. Write as a single term: $\cos\left(\frac{\pi}{4} + \theta\right) + \cos\left(\frac{\pi}{4} - \theta\right)$.

We use the formulas for the cosine of sums and differences:

$$\begin{aligned}\cos\left(\frac{\pi}{4} + \theta\right) + \cos\left(\frac{\pi}{4} - \theta\right) &= \cos\left(\frac{\pi}{4}\right)\cos(\theta) - \sin\left(\frac{\pi}{4}\right)\sin(\theta) + \cos\left(\frac{\pi}{4}\right)\cos(\theta) + \sin\left(\frac{\pi}{4}\right)\sin(\theta) \\ &= \frac{1}{\sqrt{2}}\cos(\theta) - \frac{1}{\sqrt{2}}\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta) \\ &= \sqrt{2}\cos(\theta).\end{aligned}$$

Therefore, $\cos\left(\frac{\pi}{4} + \theta\right) + \cos\left(\frac{\pi}{4} - \theta\right) = \sqrt{2}\cos(\theta)$.

6. Write as a single term: $\sin(25°)\cos(10°) - \cos(25°)\sin(10°)$.

Begin with the formula $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$, and let $\alpha = 25°$ and $\beta = 10°$.

$$\begin{aligned}\sin(25°)\cos(10°) - \cos(25°)\sin(10°) &= \sin(25° - 10°) \\ &= \sin(15°).\end{aligned}$$

7. Write as a single term: $\cos(2x)\cos(x) + \sin(2x)\sin(x)$.

Begin with the formula $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, and let $\alpha = 2x$ and $\beta = x$.

$$\begin{aligned}\cos(2x)\cos(x) + \sin(2x)\sin(x) &= \cos(2x - x) \\ &= \cos(x)\end{aligned}$$

8. Write as a single term: $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)}$, where $\cos(\alpha) \neq 0$ and $\cos(\beta) \neq 0$.

Begin with the formulas for the sine of the sum and difference:

$$\begin{aligned}\frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\cos(\alpha)\cos(\beta)} &= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} \\ &= \frac{2\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} \\ &= \frac{2\sin(\alpha)}{\cos(\alpha)} \\ &= 2\tan(\alpha).\end{aligned}$$

9. **Prove that $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$ for all values of θ.**

***PROOF:* Let θ be any real number. Then, from the formula for the cosine of a sum,**

$$\begin{aligned}\cos\left(\frac{3\pi}{2}+\theta\right) &= \cos\left(\frac{3\pi}{2}\right)\cos(\theta)-\sin\left(\frac{3\pi}{2}\right)\cdot\sin(\theta)\\ &= 0\cdot\cos(\theta)-(-1)\sin(\theta)\\ &= \sin(\theta).\end{aligned}$$

***Therefore,* $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$ *for all values of* θ.**

10. **Prove that $\cos(\pi-\theta)=-\cos(\theta)$ for all values of θ.**

***PROOF:* Let θ be any real number. Then, from the formula for the cosine of a difference,**

$$\begin{aligned}\cos(\pi-\theta) &= \cos(\pi)\cos(\theta)+\sin(\pi)\sin(\theta)\\ &= -\cos(\theta).\end{aligned}$$

***Therefore,* $\cos(\pi-\theta)=-\cos(\theta)$ *for all real numbers* θ.**